Really Natural Linear Indexed Type Checking

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Abstract

Recent works have shown the power of *linear indexed type systems* for capturing complex safety properties. These systems combine *linear type systems* with a language of *indices* that appear in the types, allowing more fine-grained analysis. For example, linear indexed types have been fruitfully applied to verify differential privacy in the *Fuzz* type system.

A natural way to enhance the expressiveness of this approach is by allowing the indices to depend on runtime information, in the spirit of dependent types. This approach is used in *DFuzz*, an extension of *Fuzz*. The *DFuzz* type system relies on an index-level language supporting real and natural number arithmetic over constants and dependent variables. Moreover, *DFuzz* uses a subtyping mechanism to semantically manipulate indices. By themselves, linearity, dependency, and subtyping each require delicate handling when performing type checking or type inference; their combination increases this challenge substantially, as the features can interact in non-trivial ways.

In this paper, we study the type-checking problem for *DFuzz*. We show how we can reduce type checking for (a simple extension of) *DFuzz* to constraint solving over a first-order theory of naturals and real numbers which, although undecidable, can often be handled in practice by standard numeric solvers.

Categories and Subject Descriptors F.3.3 [*Studies of Program Constructs*]: Type structure

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1. Introduction

Linear indexed type systems have been used to ensure safety properties of programs with respect to different kinds of resources; examples include usage analysis [24, 25], implicit complexity [4, 5, 14], sensitivity analysis [10, 23], automatic timing analysis [12, 13], and more. Linear indexed types use a type-level *index language* to describe resources and *linear types* to reason about the program's resource usage in a compositional way.

A limitation of current analysis techniques for such systems is that resource usage is inferred independently of the control flow of a program—e.g. the typing rule for branching usually approximates resources by taking the maximal usage of one of the branches, and recursion imposes even greater restrictions. To improve this scenario, some authors have proposed extending such systems with dependent types, using type indices to capture both resource usage and the *size information* of a program's input. This significantly enriches the resulting analysis by allowing resource usage to depend on runtime information. Linear dependent type systems have been used in several domains, including implicit complexity [4, 16] and sensitivity analysis [10].

Of course, there is a price to be paid for the increase in expressiveness: type checking and type inference become inevitably more complex. In linear indexed type systems, these tasks are often done in two stages: a standard Hindley-Milner-like pass, followed by a constraint-solving procedure. In some cases, the generated constraints can be solved automatically by using custom algorithms [17] or off-the-shelf SMT solvers [7, 13]. However, the constraints are specific to the index language, and richer index languages often lead to more complex constraints.

Type-checking *DFuzz*

In this paper we will focus on the type-checking problem for a particular programming language with linear dependent types, *DFuzz*. Reed and Pierce [23] recently proposed the *Fuzz* programming language, where linear indexed types are used to reason about sensitivity of programs in the context of differential privacy; the *sensitivity* of a function measures the distance between outputs on nearby inputs. In this setting, type checking and inference correspond to sensitivity analysis.

Fuzz uses real numbers as indices for the linear types. Then addition and multiplication of the indices will produce an upper bound on the sensitivity of the program. This approach gives a simple but effective sensitivity static analysis. Indeed, as shown by D'Antoni et al. [7], type-checking for *Fuzz* programs can be performed efficiently by using an SMT solver to discharge the numeric proof obligations arising from the type system. Moreover, the same approach works for type inference, which infers the minimal sensitivity of a function.

While *Fuzz* works well on a variety of simple programs, it has a fundamental limitation: sensitivity information cannot depend on runtime information, such as the size of a data structure. To get around this problem, Gaboardi et al. [10] introduced *DFuzz*, an extension of *Fuzz* with a limited form of dependent types.

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The index language in *DFuzz* combines information about the *size* of data structures with information about the *sensitivity* of functions. Technically, this is achieved by considering an index language with index variables ranging over integers (to refer to runtime sizes) and reals (to refer to runtime sensitivities). This richer index language, combined with dependent pattern-matching and subtyping, achieves increased expressiveness in the analysis, providing sensitivity bounds beyond *Fuzz*'s capabilities.

However, adding variables to the index language has a significant impact on the difficulty of type checking. Concretely, since the index language also supports addition and multiplication, index terms are now *polynomials* over the index variables. Instead of constraints between real constants like in *Fuzz*, type checking constraints in *DFuzz* may involve general polynomials.

A natural first approach is to try to extend the algorithm proposed by D'Antoni et al. [7] to work with the new index language by simply generating additional constraints when dealing with the new language constructs. This would be similar in spirit to the work of Dal Lago et al. [6] for type inference for dℓPCF, a linear dependent type system for complexity analysis. A crucial difference between that setting and *DFuzz* is that the index language of dℓPCF can be extended by arbitrary (computable) functions. This makes the approach to type inference for dℓPCF proposed by Dal Lago and Petit the most natural, since such functions can be used as direct solutions to some of the introduced constraints.

However, such an approach does not work as well for *DFuzz*, which opts for a much smaller index language. While it may be possible to extend *DFuzz*'s index language with general functions, we opt to keep the index language simple. Instead, since the type system of *DFuzz* also supports subtyping, we consider a different approach inspired by techniques from the literature on subtyping [21] and on constraint based type-inference approaches [15, 19, 22].

The main idea is to type-check a program by inferring some set of sensitivities for it, and then testing whether the resulting type is a subtype of the desired type. To obtain completeness (relative to checking the subtype), one must ensure that the inferred sensitivities are the "best" possible for that term. Unfortunately, the *DFuzz* index language is not rich enough for expressing such sensitivities. For instance, some cases require taking the maximum of two sensitivity expressions, something that cannot be done in the language of polynomials. We solve this problem by extending the index language with three syntactic constructs, resulting in a new type system that we name *EDFuzz*. This new system has meta-theoretic properties that are similar to those of *DFuzz*, but also simplifies the search for minimal sensitivities. Using these new constructs, we design a sensitivity-inference algorithm for *EDFuzz* which we show sound and complete, modulo constraint resolution.

We now face the problem of solving the constraints generated by our algorithm. First, we show how to compile the constraints generated by the algorithmic systems to constraints in the first-order theory over mixed integers and reals. This way, we can still use a numeric solver without resorting to custom symbolic resolution. Unfortunately, the presence of universal quantification over natural numbers in the constraints leads to undecidability of constraint solving; we show that *DFuzz* type-checking is undecidable, by reduction from Hilbert's tenth problem, a standard undecidable problem.

While this result shows that we can't have a terminating type-checker that is both sound and complete, not everything is lost. We first show that by approximating the constraints, we obtain a sound and computable method to type-check *EDFuzz* programs. We show that this procedure can successfully type-check a fragment of *EDFuzz* which we call *UDFuzz*; almost all of the examples proposed by Gaboardi et al. [10] belong to this class. Of course, *UDFuzz* is a strict subset of *EDFuzz*, and it is not hard to come up with well-typed programs in *EDFuzz* that are invalid under *UDFuzz*.

Finally, we present a constraint simplification procedure that can significantly reduce the complexity of our translated constraints (measured by the number of alternating quantifiers), even when checking full *EDFuzz*.

Contributions

We briefly overview the *DFuzz* programming language in Section 2, to move to an informal exposition of the main challenges involved in Section 3. Then, we present the main contributions of the paper:

- *EDFuzz*: an extension of *DFuzz* with a more expressive sensitivity language that allows to type programs with more precise types (Section 4);
- a sound and complete algorithm that reduces type checking and inference in *EDFuzz* to constraint solving over the first-order theory of $\mathbb{N}$ and $\mathbb{R}$ (Section 5 and Section 6);
- a proof of undecidability of type checking in *DFuzz* (and *EDFuzz*) (Section 7);
- a sound translation from the previous type-checking constraints to the first-order theory of the real numbers, a decidable theory (Section 8.1); and
- a simplification procedure to make the constraints more amenable to automatic solving (Section 8.2).

2. The *DFuzz* System

DFuzz [10] is a type system for verifying differential privacy. While the precise application of *DFuzz* is somewhat beyond the scope of this paper, at a high level, *DFuzz* is a system for checking function *sensitivity*. Given a notion of *distance* between values, a function f is said to be k-*sensitive* for some number k if $dist(f(x), f(y)) \leq k \cdot dist(x, y)$. Sensitivities are expressed by the index language in a linear indexed type system; let us begin by presenting *DFuzz* in some detail before discussing the type-checking challenges.

2.1 Syntax and Types

DFuzz is an extension of PCF with indexed linear types. Indices consist of numeric constants; index-level variables, which range over *sizes* (natural numbers) or *sensitivities* (positive reals extended with ∞, denoted $\mathbb{S}$); and addition and multiplication of indices. The full syntax for *DFuzz*, including the types, terms, and the index language, is shown in Figure 1. We take a brief tour through the term language.

- Abstraction and application for index variables are captured by the $\Lambda i : \kappa.e$ and $e[R]$ terms, with κ representing the kind for i. We refer to variables of natural number kind as *size variables*, while variables of real number kind are *sensitivity variables*.
- Singleton types $\mathbb{N}[S]$ and $\mathbb{R}[R]$ are used to related type-level sizes and sensitivities with term-level sizes and sensitivities.
- Dependent pattern matching over $\mathbb{N}[S]$ types is captured by the **case** construction.
- Linear types indexed by R are written $!_R\sigma \multimap \tau$.
- Variable environments Γ carry an additional annotation for assignments $x :_{[R]} \sigma$, representing the current sensitivity R for the variable x.
- Index variable environments ϕ specify the kinding of index variables.
- Constraint environments Φ store assumptions introduced under dependent pattern matching. Often, we will think of a constraint environment as the conjunction of its constraints.

$$
\begin{array}{llll}
\kappa & ::= & \mathrm{r} \mid \mathrm{n} & \text{(kinds)}\\
\mathbb{S} & ::= & \mathbb{R}^{\geq 0} \cup \{\infty\} & \text{(extended positive reals)}\\
S & ::= & i \mid 0 \mid S+1 & \text{(sizes)}\\
R & ::= & \mathbb{S} \mid i \mid S \mid R+R \mid R \cdot R & \text{(sensitivities)}\\
\sigma, \tau & ::= & \mathbb{R} \mid \mathbb{R}[R] \mid \mathbb{N}[S] \mid !_R \sigma \multimap \tau & \text{(types)}\\
& \mid & \forall i : \kappa.\ \sigma \mid \sigma \otimes \tau \mid \sigma \,\&\, \tau & \\
e & ::= & x \mid \mathbb{N} \mid \mathbf{s}\ e \mid \mathbb{R}^{\geq 0} \mid \mathbf{fix}\ (x : \sigma).e & \text{(expressions)}\\
& \mid & \lambda x :_{[R]} \sigma.e \mid e_1\ e_2 & \\
& \mid & \Lambda i : \kappa.\ e \mid e[R] & \\
& \mid & \langle e_1, e_2 \rangle \mid \pi_i\ e & \\
& \mid & (e_1, e_2) \mid \mathbf{let}\ (x, y) = e\ \mathbf{in}\ e' & \\
& \mid & \mathbf{case}\, e\, \mathbf{of}\ 0 \Rightarrow e_0 \mid n_{[i]} + 1 \Rightarrow e_s & \\
\Gamma, \Delta & ::= & \emptyset \mid \Gamma, x :_{[R]} \sigma & \text{(environments)}\\
\phi, \psi & ::= & \emptyset \mid \phi, i : \kappa & \text{(sens. environments)}\\
\Phi, \Psi & ::= & \top \mid \Phi, S = 0 \mid \Phi, S = i+1 & \text{(constraints)}
\end{array}
$$

Figure 1. *DFuzz* Types and Expressions

$$\frac{}{\phi; \Phi \models \sigma \sqsubseteq \sigma}\ \sqsubseteq\text{-Refl}$$

$$\frac{\phi; \Phi \models \sigma' \sqsubseteq \sigma \qquad \phi; \Phi \models \tau \sqsubseteq \tau'}{\phi; \Phi \models \sigma \,\&\, \tau \sqsubseteq \sigma' \,\&\, \tau'}\ (\sqsubseteq . \&)$$

$$\frac{\phi; \Phi \models \sigma \sqsubseteq \sigma' \qquad \phi; \Phi \models \tau \sqsubseteq \tau'}{\phi; \Phi \models \sigma \otimes \tau \sqsubseteq \sigma' \otimes \tau'}\ (\sqsubseteq . \otimes)$$

$$\frac{\models \forall \phi.\ (\Phi \Rightarrow R \leq R') \quad \phi; \Phi \models \sigma' \sqsubseteq \sigma \qquad \phi; \Phi \models \tau \sqsubseteq \tau'}{\phi; \Phi \models !_R \sigma \multimap \tau \sqsubseteq !_{R'} \sigma' \multimap \tau'}\ (\sqsubseteq . \multimap)$$

$$\frac{\phi, i : \kappa; \Phi \models \sigma \sqsubseteq \tau \qquad i \text{ fresh in } \phi}{\phi; \Phi \models \forall i : \kappa.\ \sigma \sqsubseteq \forall i : \kappa.\ \tau}\ (\sqsubseteq . \forall)$$

$$\frac{\forall (x :_{[R_i]} \sigma_i, x :_{[R'_i]} \sigma_i) \in (\Gamma, \Delta) \quad \mathrm{dom}(\Delta) \subseteq \mathrm{dom}(\Gamma) \qquad \models \forall \phi.\ (\Phi \Rightarrow R_i \geq R'_i)}{\phi; \Phi \models \Gamma \sqsubseteq \Delta}\ \sqsubseteq\text{-Env}$$

Figure 2. *DFuzz* Subtyping Relation

2.2 Environment Operations

As is the case for many linear type systems, *DFuzz* defines operations on variable environments. Precisely, two environments Γ, Δ can be combined with addition, and a single environment Γ can multiplied by a sensitivity (a sort of environment scaling). Throughout, we will write $\mathrm{dom}(\Gamma)$ for Γ's domain.

We define environment multiplication $R \cdot \Gamma$ as the operation taking every element $x_i :_{[r_i]} \sigma_i$ of Γ to $x_i :_{[R \cdot r_i]} \sigma_i$. Environment addition is defined iff all the common assignments of Γ, Δ map to the same type, that is to say, forall x_i in $\mathrm{dom}(\Gamma) \cap \mathrm{dom}(\Delta)$, $(x_i :_{[R_i]} \sigma_i) \in \Gamma \iff (x_i :_{[S_i]} \sigma_i) \in \Delta$. In such case:

$$
\begin{array}{rcll}
\Gamma + \Delta & = & \{x_i :_{[R_i + S_i]} \sigma & \mid x_i \in \mathrm{dom}(\Gamma) \cap \mathrm{dom}(\Delta)\}\\
& \cup & \{x_j :_{[R_j]} \sigma_j & \mid x_j \in \mathrm{dom}(\Gamma) - \mathrm{dom}(\Delta)\}\\
& \cup & \{x_k :_{[R_k]} \sigma_k & \mid x_k \in \mathrm{dom}(\Delta) - \mathrm{dom}(\Gamma)\}
\end{array}
$$

2.3 Subtyping

DFuzz has a notion of subtyping, which intuitively corresponds to a standard property of function sensitivity: a k-sensitive function is also k'-sensitive for all $k' \geq k$. Furthermore, subtyping in *DFuzz* is the mechanism that allows types to use information from the constraint environment; in this use, subtyping allows a form of type coercion. We consider here a slightly simpler definition of subtyping than the one used in Gaboardi et al. [10]. In the environments we requires subtyping to preserve the internal type. This slight modification will allow us to simplify some rules of the type-checking algorithm.

The semantics of the subtying relation is defined by interpreting sensitivity expressions as functions that produce sensitivity values. Formally, let R be a sensitivity expression, well-typed under environment ϕ, and ρ a suitable variable *valuation* (i.e., a function that maps each variable $x : \kappa$ in ϕ to an element of $[\![\kappa]\!]$, with $[\![\mathrm{n}]\!] = \mathbb{N}$ and $[\![\mathrm{r}]\!] = \mathbb{S}$). We then define $[\![R]\!]_\rho$ as follows:

$$
\begin{array}{rcll}
[\![0]\!]_\rho & := & 0 & \\
[\![S+1]\!]_\rho & := & [\![S]\!]_\rho + 1 & \\
[\![i]\!]_\rho & := & \rho(i) & i \text{ a variable}\\
[\![r]\!]_\rho & := & r & r \text{ a constant}\\
[\![R_1 + R_2]\!]_\rho & := & [\![R_1]\!]_\rho + [\![R_2]\!]_\rho & \\
[\![R_1 \cdot R_2]\!]_\rho & := & [\![R_1]\!]_\rho \cdot [\![R_2]\!]_\rho &
\end{array}
$$

Then, the standard ordering $\geq$ on $\mathbb{S}$ (i.e., the positive real numbers with a maximal element ∞) induces an ordering on index terms, which we can then extend to a subtype relation $\sqsubseteq$ on types and environments; the rules can be found in Figure 2. Note that checking happens under the current constraint environment Φ, so subtyping may use information recovered from a dependent match.

The leaves of the subtype derivation are either equalities that are consequences of the constraint environment Φ, or assertions $\phi \models (\Phi \Rightarrow R_1 \geq R_2)$. These are defined logically as

$$\forall \rho.(\mathsf{dom}(\rho) = \phi \wedge \rho(\Phi)) \Rightarrow [\![R_1]\!]_\rho \geq [\![R_2]\!]_\rho,$$

where the quantification is over all well-kinded substitutions ρ for variables specified by ϕ satisfying the constraints Φ.

2.4 Typing

Typing judgments for *DFuzz* are of the form

$$\phi; \Phi \mid \Gamma \vdash e : \sigma$$

meaning that term e has type σ under environments ϕ and Γ and constraints Φ; full rules are shown in Figure 3.

We highlight here just the most complex rule, the dependent pattern matching rule $(\mathbb{N}\ E)$, which allows each branch to be typed under different assumptions on the type $\mathbb{N}[S]$ of the scrutinee (e). The left branch e_0 is typed under the assumption $S = 0$, while the right branch e_s is typed under the assumption $S = i + 1$ for some i. Indeed, this rule is useful for capturing programs whose sensitivity depends on the number of iterations or number of input elements; combined with the fix rule (Fix), these features enable programs that iterate depending on a runtime parameter while still reasoning about the number of iterations. Readers interested in more details can consult Gaboardi et al. [10]; we follow their presentation closely except for a few points, which we detail in the Appendix.

2.5 Examples

We close the overview of *DFuzz* with some examples. The first example is multiplication. Usually, multiplication cannot be assigned a type as is not sensitive for any k. However, thanks to dependent types we can introduce a multiplication primitive with type:

$$\times : \forall R_1 : \mathrm{r}.\forall R_2 : \mathrm{r}.!_{R_1} \mathbb{R}[R_1] \multimap !_{R_2} \mathbb{R}[R_2] \multimap \mathbb{R}[R_1 \cdot R_2]$$

A function that adds ϵ noise to the output has type:

$$\text{add\_noise} : \forall \epsilon : \mathrm{r}.!_\epsilon \mathbb{R} \multimap \bigcirc \mathbb{R}$$

where $\bigcirc\mathbb{R}$ is the type of probability distributions over $\mathbb{R}$.

$$\frac{\phi;\Phi \mid \Delta \vdash e : \sigma \qquad \phi;\Phi \models \Gamma \sqsubseteq \Delta}{\phi;\Phi \mid \Gamma \vdash e : \sigma} \ (\sqsubseteq .\mathrm{L}) \qquad \frac{\phi;\Phi \mid \Gamma \vdash e : \sigma \qquad \phi;\Phi \models \sigma \sqsubseteq \tau}{\phi;\Phi \mid \Gamma \vdash e : \tau} \ (\sqsubseteq .\mathrm{R})$$

$$\frac{r \in \mathbb{R}}{\phi;\Phi \mid \Gamma \vdash r : \mathbb{R}} \ (\mathrm{Const}_{\mathbb{R}}) \qquad \frac{n = [\![S]\!]}{\phi;\Phi \mid \Gamma \vdash n : \mathbb{N}[S]} \ (\mathrm{Const}_{\mathbb{N}})$$

$$\frac{}{\phi;\Phi \mid \Gamma, x :_{[1]} \sigma \vdash x : \sigma} \ (\mathrm{Var}) \qquad \frac{\phi;\Phi \mid \Gamma, x :_{[\infty]} \sigma \vdash e : \sigma}{\phi;\Phi \mid \infty \cdot \Gamma \vdash \mathbf{fix}\,(x : \sigma).e : \sigma} \ (\mathrm{Fix})$$

$$\frac{\phi;\Phi \mid \Gamma, x :_{[R]} \sigma \vdash e : \tau}{\phi;\Phi \mid \Gamma \vdash \lambda x :_{[R]} \sigma.e : !_R\sigma \multimap \tau} \ (\multimap I) \qquad \frac{\phi;\Phi \mid \Gamma \vdash e_1 : !_R\sigma \multimap \tau \qquad \phi;\Phi \mid \Delta \vdash e_2 : \sigma}{\phi;\Phi \mid \Gamma + R \cdot \Delta \vdash e_1\ e_2 : \tau} \ (\multimap E)$$

$$\frac{\phi, i : \kappa;\Phi \mid \Gamma \vdash e : \sigma \qquad i \text{ fresh in } \Phi, \Gamma}{\phi;\Phi \mid \Gamma \vdash \Lambda i : \kappa.\ e : \forall i : \kappa.\ \sigma} \ (\forall I) \qquad \frac{\phi;\Phi \mid \Gamma \vdash e : \forall i : \kappa.\ \sigma \qquad \phi \models S : \kappa}{\phi;\Phi \mid \Gamma \vdash e[S] : \sigma[S/i]} \ (\forall E)$$

$$\frac{\phi;\Phi \mid \Gamma_1 \vdash e_1 : \sigma \qquad \phi;\Phi \mid \Gamma_2 \vdash e_2 : \tau}{\phi;\Phi \mid \Gamma_1 + \Gamma_2 \vdash (e_1, e_2) : \sigma \otimes \tau} \ (\otimes I) \qquad \frac{\phi;\Phi \mid \Delta \vdash e : \sigma \otimes \tau \qquad \phi;\Phi \mid \Gamma, x :_{[R]} \sigma, y :_{[R]} \tau \vdash e' : \mu}{\phi;\Phi \mid \Gamma + R \cdot \Delta \vdash \mathbf{let}\,(x, y) = e\,\mathbf{in}\,e' : \mu} \ (\otimes E)$$

$$\frac{\phi;\Phi \mid \Gamma \vdash e_1 : \sigma \qquad \phi;\Phi \mid \Gamma \vdash e_2 : \tau}{\phi;\Phi \mid \Gamma \vdash \langle e_1, e_2 \rangle : \sigma \,\&\, \tau} \ (\&\, I) \qquad \frac{\phi;\Phi \mid \Gamma \vdash e : \sigma_1 \,\&\, \sigma_2}{\phi;\Phi \mid \Gamma \vdash \pi_i\ e : \sigma_i} \ (\&\, E)$$

$$\frac{\phi;\Phi \mid \Gamma \vdash e : \mathbb{N}[S]}{\phi;\Phi \mid \Gamma \vdash \mathbf{s}\ e : \mathbb{N}[S+1]} \ (\mathrm{S}\ I) \qquad \frac{\begin{array}{c}\phi;\Phi \mid \Delta \vdash e : \mathbb{N}[S] \qquad \phi;\Phi, S = 0 \mid \Gamma \vdash e_0 : \sigma \\ \phi, i : \mathrm{n};\Phi, S = i+1 \mid \Gamma, n :_{[R]} \mathbb{N}[i] \vdash e_s : \sigma \qquad i \text{ fresh in } \phi\end{array}}{\phi;\Phi \mid \Gamma + R \cdot \Delta \vdash \mathbf{case}\,e\,\mathbf{return}\,\sigma\,\mathbf{of}\,0 \Rightarrow e_0 \mid n_{[i]} + 1 \Rightarrow e_s : \sigma} \ (\mathbb{N}\ E)$$

Figure 3. *DFuzz* Typing Rules

Functions sensitive on number of iterations or size of the input are similarly typed. A function that adds noise i times to an input is:

$$\mathsf{iNoise} : \forall i : \mathrm{n}, \forall \epsilon : \mathrm{r}.!_\infty \mathbb{N}[i] \multimap !_\infty \mathbb{R}[\epsilon] \multimap !_{i \cdot \epsilon} \mathbb{R} \multimap \mathbb{R}$$

3. The Challenge of Type-checking Linear Dependent Types

Type-checking a language with linear indexed types presents several challenges, which are only compounded when dependent types and subtyping are added to the mix. In this section, we take a closer look at these challenges.

3.1 To Split, or not to Split?

The first problem we face is due to linearity. Given a term and an environment, we need a way to "split" the environment into appropriate sub-environment that can be used in the recursive calls to type check subterms.

Automatically inferring the right environments in our setting is difficult, due to the index language for *DFuzz*. Indeed, index terms are polynomials over index variables, which may range over the reals or the naturals. For instance, we may know that a particular variable x has sensitivity $i^2 \cdot j^2 + 3$ in our environment. However, it is not clear how to split such sensitivity information between two environments that share the variable x. In fact, as we will show below, in general it is not always possible to find a split. One might hope to simplify the type-checking task by requiring the programmer to provide a few type annotations, like in non-linear type systems. Unfortunately, this approach is impractical for the splitting problem because naively, the annotations must describe the split for every variable binding in the context!

To better understand this obstacle, let us consider two general approaches to type-checking linear type systems, which we call the *top-down* and *bottom-up* strategies.

The Downfall of Top-Down

For the type-checking problem, suppose we are given the environment Γ, a term e, and a purported type σ. The goal is to decide if $\Gamma \vdash e : \sigma$ is derivable. The *top-down* strategy takes a context and a term, and attempts to partition the context and recursively type the subterms of e.

The main difficulty of this approach centers around splitting the environment, a problem that is most clear in the application rule. Here is a simplified version:

$$\frac{\Gamma \vdash f : !_R\sigma \multimap \tau \qquad \Delta \vdash e : \sigma}{\Gamma + R \cdot \Delta \vdash f\ e : \tau}$$

So given a type-checking problem $\Sigma \vdash f\ e : \sigma'$ our first difficulty is to pick R, Γ, and Δ such that $\Sigma = \Gamma + R \cdot \Delta$. We could try to guess R, but unfortunately it may depend on the choice of Γ. Since our index language contains the real numbers, the number of possible splittings isn't even finite.

A natural idea is to delay the choice of this split. For instance, we may create a placeholder variable R and placeholder environments Γ', Δ', asserting $\Sigma = \Gamma' + R \cdot \Delta'$ and recursively type-checking f and e. After reaching the leaves of the derivation, we would have a set of constraints whose satisfiability would imply that the program type-checks.

Unfortunately, the constraints seem difficult to solve due to the syntactical nature of our indices. In other words, the "placeholder variables" are really *meta-variables* that range over index terms, which could potentially depend on bound index variables. In order to prove soundness of such a system with respect to the formal typing system, the solver must return success *only* if there is a solution where all the meta-variables can be instantiated to an index term—a syntactic object. This is at odds with the way most solvers work—*semantically*—finding arbitrary solutions over their domain.

It is not clear how to solve these existential constraints automatically for the specific index language of *DFuzz*.

The Rise of Bottom-Up?

A different approach is a *bottom-up* strategy: suppose we are again given an environment Γ, a term e, and a type σ, and we want to check if $\Gamma \vdash e : \sigma$ is derivable. The main idea is to avoid splitting environments by calculating the minimal sensitivities needed for typing each subexpression. For each typing rule, these minimal sensitivities can be combined to find the resulting minimal sensitivities for e. Once this is done, we just need to check whether these optimal sensitivities are compatible with Γ and σ via subtyping.

Let's consider how this works in more detail by analyzing a few important cases. At the base case, we type-check variables in a minimal context (that is, empty but for the variable) an assigning it the minimal sensitivity required:

$$\frac{}{x :_{[1]} \sigma \vdash x : \sigma}$$

Recall that we have weakening on the left so can add non-occurring variables to the context later.

Now, the key benefit of the bottom-up approach becomes evident in the application rule: we can completely avoid the splitting problem. When faced with a type-checking instance $\Sigma \vdash f\ e : \sigma$, we recursively find optimal Γ, R, and Δ for checking f and e; then, checking that $\Sigma \sqsubseteq \Gamma + R \cdot \Delta$ suffices.

Unfortunately, things don't look so easy in the additive rules. Let's examine the introduction rule for $\&$:

$$\frac{\Gamma \vdash e_1 : \sigma_1 \qquad \Gamma \vdash e_2 : \sigma_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \sigma_1 \ \&\ \sigma_2}$$

This rule forces both environments to have the same sensitivities, but the bottom-up idea may infer different environments for each expression:

$$\frac{\Gamma_1 \vdash e_1 : \sigma_1 \qquad \Gamma_2 \vdash e_2 : \sigma_2}{\Sigma? \vdash \langle e_1, e_2 \rangle : \sigma_1 \ \&\ \sigma_2}$$

Now we need to guess a best environment $\Sigma?$, but the *DFuzz* sensitivity language is too weak to express this value. For instance, if we consider sensitivity expressions r^2 and r depending on a sensitivity variable r, we can show that there is no minimal polynomial upper bound for them under the point-wise order on polynomials[1].

To maintain the minimality invariant, we can extend the sensitivity language with a new syntactic construct $\mathbf{max}(R_1, R_2)$ for type-inference purposes only, which should denote the maximum of two sensitivity values. We could then safely set $\Sigma? := \mathbf{max}(\Gamma_1, \Gamma_2)$, where the expression combines sensitivities for the bindings on both environments as expected.

However, there is a problem with this approach: the resulting algorithm is not sound with respect to the original type system, because it allows more terms to be typed even when sensitivities in the final type do not mention the new construct! To see this, assume that our algorithm produces a derivation $\Gamma' \vdash e : \sigma'$ using extended sensitivities. Now, soundness amounts to showing that for all Γ, σ mentioning only standard sensitivities such that $\Gamma \sqsubseteq \Gamma'$ and $\sigma' \sqsubseteq \sigma$, there exists a typing derivation $\Gamma \vdash e : \sigma$ that uses only the original sensitivity language. Let's try to sketch how this proof would work by restricting our attention to a particular instance of the application rule:

$$\frac{\phi; \emptyset \mid \emptyset \vdash f : !_{R_f} \sigma \multimap \tau \qquad \phi; \emptyset \mid x :_{[\hat{R}_x]} \mu \vdash e : \sigma}{\phi; \emptyset \mid x :_{[R_f \cdot \hat{R}_x]} \mu \vdash f\ e : \tau}$$

[1] Indeed, it can be seen that *DFuzz* does not possess minimal types. Refer to the Appendix for a more detailed proof.

where $\hat{R}_x$ is an extended sensitivity expression. By induction, we know that for all standard sensitivity expressions R_x such that $R_x \geq \hat{R}_x$, we can obtain a standard derivation $x :_{[R_x]} \mu \vdash e : \sigma$. We also have standard R_{xf} such that $R_{xf} \geq R_f \cdot \hat{R}_x$. Thus, all we need to do is to calculate from R_f, R_{xf} standard sensitivities R'_f, R'_x to be able to apply both induction hypotheses. The following result shows that this is not always possible.

Lemma 1. *Given standard sensitivities expressions R_{xf}, R_f and an extended sensitivity expression $\hat{R}_x$ such that $R_{xf} \geq R_f \cdot \hat{R}_x$, it is not the case that one can always find standard R'_f, R'_x such that $R_{xf} \geq R'_f \cdot R'_x \wedge R'_f \geq R_f \wedge R'_x \geq \hat{R}_x$.*

Proof. Take $R_{xf} = r^2 + 1$, $R_f = r$ and $\hat{R}_x = \mathbf{max}(2, r)$. As we can see, we have $r^2 + 1 \geq r \cdot \mathbf{max}(2, r)$, with a strict equality iff $r = 1$. Suppose there exist standard sensitivity expressions R'_f, R'_x such that they satisfy the statement. Because $R'_f \geq r$ and $R'_x \geq \mathbf{max}(2, r)$, we know by asymptotic analysis that the degree of R'_f and R'_x must be at least 1. Furthermore, because $r^2 + 1 \geq R'_f \cdot R'_x$, their degree must be exactly 1, with leading coefficient equal to 1. Write $R'_f = r + a$ and $R'_x = r + b$, where a, b are positive constants. The lower bound on R'_x implies $b \geq 2$. For $r = 1$, we have $R'_f \cdot R'_x \geq 3a + 3 \geq 3$. However, the lower and upper bounds for $R'_f \cdot R'_x$ coincide at that point, forcing $R'_f \cdot R'_x = 2$; contradiction. Thus, no such R'_f, R'_x can exist. $\square$

It is not hard to adapt the above into a counterexample for the soundness of the algorithm with respect to the original system. However, we can recover soundness by extending the sensitivity language for the basic typing rules as well.

3.2 Avoiding the Avoidance Problem

After the addition of least upper bounds for sensitivities, the bottom-up approach is in a good working state for the basic system. However, other constructs in the language introduce further challenges. In particular, let's examine a simple version of the abstraction rule for sensitivity variables:

$$\frac{\phi, i : \kappa \mid \Gamma \vdash e : \sigma \qquad i \text{ fresh in } \Gamma}{\phi \mid \Gamma \vdash \Lambda i : \kappa.\ e : \forall i : \kappa.\ \sigma}$$

When this rule is interpreted in a top-down approach, usually no problem arises; we would just introduce the new sensitivity variable and proceed with type checking.

However, when the typechecking direction is reversed, we hit a version of the avoidance problem [8, 11, 18]. The avoidance problem usually appears in slightly different scenarios related to existential types, and could be informally stated as finding a best type free of a particular variable. In our case, we must find the "best" Γ free of i. It may not be obvious how i could have been propagated to Γ, but indeed, a function f in e could have a type like as $!_i\sigma \multimap \tau$, and applying f will introduce i into the environment in the bottom-up approach.

Fortunately, in our setting, we can easily solve the avoidance problem by further extending the sensitivity language. The "best" way of freeing a sensitivity expression R of a variable i is to take the supremum of R over all possible values of i, which we denote by $\mathbf{sup}(i, R)$[2]. Then, the minimal environment is $\mathbf{sup}(i, \Gamma)$, where the supremum is extended to each binding in the environment.

[2] Contrary to $\mathbf{max}(-, -)$, it would have been possible to define this construct as a function over sensitivity expressions, without the need to extend their syntax. This would still be true even after introducing index-level case sensitivity expression for analyzing dependent pattern matching. As the translation is somewhat intricate and leads to more complex constraints, we chose to add it directly to the syntax of sensitivity expressions.

3.3 Undependable Dependencies

The last case to consider in our informal overview is **case**, also referred as dependent pattern matching.

The dependent pattern matching can be considered as a special case of the two previous difficulties. Like the least upper bound, we must compute a least upper bound of the resources used in two branches. However, now the information coming from the successor branch may also contain sensitivities depending on the newly introduced refinement variable, which cannot occur in the upper bound; similar to the avoidance problem we just discussed. On top of that, information coming from both sides is conditional on the particular refinements induced by the match, so any new sensitivity information that we propagate cannot really depend on the refinements.

We now face a choice: we can introduce refinement types over sensitivity and size variables of the form $\{\sigma \mid P(\vec{i})\}$, which would allow us to express the sensitivty inference for **case** in term of the least upper bound and supremum operations. However, we take a simpler path and add a conditional operator on natural number expressions S, $\mathbf{case}(S, R_0, i, R_s)$, interpreted as R_0 if S is 0 or $R_s[i \mapsto S-1]$ if $S \geq 1$.

In the next sections we proceed to formally introduce the extended sensitivities and its semantics; we discuss the type-checking algorithm, which depends on solving inequality constraints over the extended sensitivities; and we study several approaches and discuss their decidability.

4. Extended *DFuzz*: *EDFuzz*

We define a conservative extension to *DFuzz*'s type system, *EDFuzz*, which is basically *DFuzz* with an extended sensitivity language for the indices. We summarize the new sensitivity terms:

- $\mathbf{max}(R_1, R_2)$ is the pointwise least upper bound of sensitivity terms R_1, R_2.
- $\mathbf{sup}(i, R)$ is the pointwise least upper bound of R over all i.
- $\mathbf{case}(S, R_0, i, R_s)$ is the conditional function on the size expression S that is valued R_0 when $S = 0$, and $R_s[i \mapsto S-1]$ when S is a strictly positive integer.

We write $\hat{R}$ for the *extended sensitivity language*, built from the standard sensitivity terms and operations and the new extended terms. The semantics of extended terms are defined as follows.

Definition 2 (Extended sensitivity semantics)**.** *For every well-kinded valuation $\phi \models \rho$ for $\phi \models R$ we have:*

$$\llbracket \mathbf{sup}(i : \kappa, \hat{R}) \rrbracket_\rho := \sup_{r \in \kappa} \{\llbracket \hat{R} \rrbracket_{\rho \cup [i=r]}\}$$

$$\llbracket \mathbf{max}(\hat{R}_1, \hat{R}_2) \rrbracket_\rho := \max(\llbracket \hat{R}_1 \rrbracket_\rho, \llbracket \hat{R}_2 \rrbracket_\rho)$$

$$\llbracket \mathbf{case}(S, \hat{R}_0, i, \hat{R}_s) \rrbracket_\rho := \begin{cases} \llbracket \hat{R}_0 \rrbracket_\rho & \textit{if} \quad \llbracket S \rrbracket_\rho = 0 \\ \llbracket \hat{R}_s \rrbracket_{\rho \cup [i=n-1]} & \textit{if} \quad \llbracket S \rrbracket_\rho = n \geq 1. \end{cases}$$

$$\llbracket \hat{R}_1 + \hat{R}_2 \rrbracket_\rho := \llbracket \hat{R}_1 \rrbracket_\rho + \llbracket \hat{R}_2 \rrbracket_\rho$$

$$\llbracket \hat{R}_1 \cdot \hat{R}_2 \rrbracket_\rho := \llbracket \hat{R}_1 \rrbracket_\rho \cdot \llbracket \hat{R}_2 \rrbracket_\rho.$$

We define analogous operations on contexts in the obvious way. For instance, if $x :_{R_1} \sigma \in \Gamma_1$ and $x :_{R_2} \sigma \in \Gamma_2$, then $x :_{\mathbf{max}(R_1, R_2)} \sigma \in \mathbf{max}(\Gamma_1, \Gamma_2)$. Context operations that take two contexts Γ_1, Γ_2 are only defined if the contexts have the same skeleton, i.e., $\Gamma_1^\bullet = \Gamma_2^\bullet$.

It is not hard to show that any derivation valid in *DFuzz* remains valid in *EDFuzz*. Furthermore, *DFuzz*'s metatheory only relies on sensitivity terms having an interpretation as total function from free variables to a real number, rather than on any specific property about the interpretation itself. The extended interpretation is total, and hence the metatheory of *DFuzz* extends to *EDFuzz*.

5. Type Checking and Inference

We present a sound and complete type checking and inference algorithm for *EDFuzz*. The algorithm assumes the existence of an oracle for deciding the subtyping relation, so in that sense our algorithm is relatively complete. We defer discussion about solving subtyping constraints to the next section.

We remind the reader that our definitions of type-checking and inference assume that a regular typing derivation—that is to say, erasing all linear types and dependent terms—for an expression is already known. This can be computed, for example, by a Hindley-Milner-style pass. Here and below, we focus on handling the sensitivities. For a type σ, we write $\overline{\sigma}$ for the type where all linear types are mapped to regular function types and all the dependently typed types are mapped to their non-dependent version. This erasure operation is extended to environments in the natural way: Given an environment Γ, we define its skeleton as $\overline{\Gamma}$, containing a list of type bindings $(x : \sigma)$, but *without* the external sensitivities (i.e., the annotation on the colon).

Definition 3 (Type Checking)**.** *Given a context Γ, a term e, a type σ, and a HM derivation $\overline{\Gamma} \vdash_H e : \overline{\sigma}$, then the type-checking problem for* EDFuzz *is to determine whether a derivation $\emptyset; \emptyset; \Gamma \vdash e : \sigma$ exists.*

In our context, type inference means inferring the sensitivity annotations in both a context and a type.

Definition 4 (Type Inference)**.** *Given a context skeleton $\overline{\Gamma}$, a term e, a regular type σ, and a HM derivation $\overline{\Gamma} \vdash_H e : \sigma$, the type-inference problem is to compute a context Γ and a type τ such that a derivation $\emptyset; \emptyset; \Gamma \vdash e : \tau$ exists and $\overline{\Gamma} = \Gamma$ and $\overline{\tau} = \sigma$.*

5.1 The Algorithm

We can fulfill both goals using an algorithm that takes as inputs a term e, an environment free of sensitivity annotations $\Gamma^\bullet$ and a refinement constraint Φ. The algorithm will output an annotated environment Δ and a type σ. We write a call to the type inference algorithm as:

$$\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Delta; \sigma.$$

Figure 4 presents the full algorithm in a judgmental style. The algorithm is based on a syntax-directed version of DFuzz that enjoys several nice properties; full technical details can be found in the Appendix. Here, we just sketch how the transformation works in the proofs of soundness and completeness.

Theorem 5 (Algorithmic Soundness)**.** *Suppose $\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Gamma; \sigma$. Then, there is a derivation of $\phi; \Phi; \Gamma \vdash e : \sigma$.*

Proof. We define two intermediate systems: The first one internalizing certain properties of weakening and a second, syntax-directed. The algorithm is a direct transcription of the syntax-directed system and soundness can be proved by induction on the number of steps. We prove soundness of the syntax-directed system by induction on the syntax-directed derivation. □

Theorem 6 (Algorithmic Completeness)**.** *Suppose $\phi; \Phi; \Gamma \vdash e : \sigma$ is derivable. Then $\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Gamma'; \sigma'$ and $\phi; \Phi \models \Gamma \sqsubseteq \Gamma' \wedge \sigma' \sqsubseteq \sigma$.*

Proof. We show that a "best" syntax-directed derivation can be build from any standard derivation by induction on the original derivation plus monotonicity and commutativity properties of the subtype relation. Completeness for the algorithm follows. □

$$\frac{}{\phi; \Phi; \Gamma^\bullet; r \Longrightarrow \mathrm{Ectx}(\Gamma^\bullet); \mathbb{R}} \ (\mathrm{Const}) \qquad \frac{n = [\![S]\!]}{\phi; \Phi; \Gamma^\bullet; n \Longrightarrow \mathrm{Ectx}(\Gamma^\bullet); \mathbb{N}[S]} \ (\mathrm{Const}_{\mathbb{N}})$$

$$\frac{}{\phi; \Phi; \Gamma^\bullet, x : \sigma; x \Longrightarrow \mathrm{Ectx}(\Gamma^\bullet), x :_{[1]} \sigma; \sigma} \ (\mathrm{Var}) \qquad \frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet, x : \sigma; e \Longrightarrow \Gamma, x :_{[R']} \sigma; \tau \\ \phi; \Phi \models R \geq R'_{\Box\uparrow}\end{array}}{\phi; \Phi; \Gamma^\bullet; \lambda(x :_{[R]} \sigma).\, e \Longrightarrow \Gamma; !_R\sigma \multimap \tau} \ (\multimap I)$$

$$\frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet; e_1 \Longrightarrow \Gamma; !_R\sigma \multimap \tau \\ \phi; \Phi; \Delta^\bullet; e_2 \Longrightarrow \Delta; \sigma' \\ \phi; \Phi \models \sigma' \sqsubseteq \sigma\end{array}}{\phi; \Phi; \Gamma^\bullet; e_1\ e_2 \Longrightarrow \Gamma + R \cdot \Delta; \tau} \ (\multimap E) \qquad \frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet, x : \sigma; e \Longrightarrow \Gamma, x :_{[R]} \sigma; \sigma' \\ \phi; \Phi \models \sigma' \sqsubseteq \sigma\end{array}}{\phi; \Phi; \Gamma^\bullet; \mathbf{fix}\, x : \sigma.\, e : \sigma \Longrightarrow \infty \cdot \Gamma; \sigma} \ (\mathrm{Fix})$$

$$\frac{\phi, i : \kappa; \Phi; \Gamma^\bullet; e \Longrightarrow \Gamma; \sigma}{\phi; \Phi; \Gamma^\bullet; \Lambda i : \kappa.\, e \Longrightarrow \mathbf{sup}(i, \Gamma); \forall i : \kappa.\, \sigma} \ (\forall I) \qquad \frac{\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Gamma; \forall i : \kappa.\, \sigma \qquad \phi \models S : \kappa}{\phi; \Phi; \Gamma^\bullet; e[S] \Longrightarrow \Gamma; \sigma[S/i]} \ (\forall E)$$

$$\frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet; e_1 \Longrightarrow \Gamma_1; \sigma_1 \\ \phi; \Phi; \Gamma^\bullet; e_2 \Longrightarrow \Gamma_2; \sigma_2\end{array}}{\phi; \Phi; \Gamma^\bullet; \langle e_1, e_2 \rangle \Longrightarrow \Gamma_1 + \Gamma_2; \sigma_1 \otimes \sigma_2} \ (\otimes I) \qquad \frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Delta; \sigma \otimes \tau \\ \phi; \Phi; \Gamma^\bullet, x : \sigma, y : \tau; e' \Longrightarrow \Gamma, x :_{[R_1]} \sigma, y :_{[R_2]} \tau; \mu\end{array}}{\phi; \Phi; \Gamma^\bullet; \mathbf{let}(x, y) = e\ \mathbf{in}\ e' \Longrightarrow \Gamma + \mathbf{max}(R_{1\Box\uparrow}, R_{2\Box\uparrow}) \cdot \Delta; \mu} \ (\otimes E)$$

$$\frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet; e_1 \Longrightarrow \Gamma_1; \sigma_1 \\ \phi; \Phi; \Gamma^\bullet; e_2 \Longrightarrow \Gamma_2; \sigma_2\end{array}}{\phi; \Phi; \Gamma^\bullet; \langle e_1, e_2 \rangle \Longrightarrow \mathbf{max}(\Gamma_1, \Gamma_2); \sigma_1 \,\&\, \sigma_2} \ (\&\ I) \qquad \frac{\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Gamma; \sigma_1 \,\&\, \sigma_2}{\phi; \Phi; \Gamma^\bullet; \pi_i e \Longrightarrow \Gamma; \sigma_i} \ (\&\ E)$$

$$\frac{\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Gamma; \mathbb{N}[S]}{\phi; \Phi; \Gamma^\bullet; \mathbf{s}\, e \Longrightarrow \Gamma; \mathbb{N}[S+1]} \ (\mathrm{S}\ I) \qquad \frac{\begin{array}{l}\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Delta; \mathbb{N}[S] \qquad \phi; \Phi, S = 0; \Gamma^\bullet; e_0 \Longrightarrow \Gamma_0; \sigma_0 \\ \phi, i : \mathrm{n}; \Phi, S = i + 1; \Gamma^\bullet, x : \mathbb{N}[i]; e_s \Longrightarrow \Gamma_s, x :_{[R']} \mathbb{N}[i]; \sigma_s \\ \phi; \Phi, S = 0 \models \sigma_0 \sqsubseteq \sigma \qquad \phi, i : \mathrm{n}; \Phi, S = i + 1 \models \sigma_s \sqsubseteq \sigma\end{array}}{\begin{array}{l}\phi; \Phi; \Gamma^\bullet; \mathbf{case}\, e\, \mathbf{return}\, \sigma\, \mathbf{of}\, 0 \mapsto e_0 \mid x_{[i]} + 1 \mapsto e_s \\ \Longrightarrow \mathbf{case}(S, \Gamma_0, i, \Gamma_s) + \mathbf{case}(S, 0, i, R'_{\Box\uparrow}) \cdot \Delta; \sigma\end{array}} \ (\mathbb{N}\ E)$$

Figure 4. Algorithmic Rules for *EDFuzz*

5.2 Removing Sensitivity Annotations

We briefly discuss the role annotations play in our algorithm. *DFuzz* programs have three different annotations: the type of the argument for lambda terms (including the sensitivity), the return type for case, and the type for fixpoints.

The sensitivity annotations ensure that inferred types are free of terms with extended sensitivities. This is useful for some optimizations on subtype checking (introduced later in the paper). However, the general encoding of subtyping checks works with full extended types, thus the sensitivity annotations can be safely omitted and the system will infer types containing extended sensitivities.

Due to technical difficulties in inferring the minimal sensitivity in the presence of higher-order functions, the argument type in functions (σ in $\lambda(x : \sigma)$) must be annotated, and we require the type of fixpoints to be annotated.

6. Constraint Solving over Mixed Reals/Naturals

The type-checking algorithm introduced in the previous section produces inequality constraints over the extended sensitivity language. While these extended sensitivity terms may appear complicated, we can translate them into formulas in the first-order theory of $\mathbb{S}$ and $\mathbb{N}$ in a sound and complete way.

While we will show in the next section that the kind of first-order formulas we generate here are in general undecidable, they can still be handled by numeric solvers providing mixed real/natural theories. Moreover, in Section 8.1 we will present a sound (although not complete) computable procedure to check the constraints.

Quantification over $\mathbb{S}$ should be interpreted as quantification over $\mathbb{R}^\infty$ with a non-negativity constraint; all the quantifiers in our target first-order theory will range over either $\mathbb{R}^\infty$ or $\mathbb{N}$. (In the next section, we will show that quantifying over just $\mathbb{R}$ and $\mathbb{N}$ is enough.)

The idea behind our translation is simple: we use a first-order formula to uniquely specify each extended sensitivity term. In other words, we define a predicate $T(R)$ for each extended sensitivity term R, such that $[\![T(R)(r)]\!]_\rho$ holds exactly when r is equal to the interpretation of R under the valuation ρ. For instance, consider the translation for $R_1 + R_2$:

$$T(R_1{+}R_2)(r) := \exists r_1\, r_2 : \mathbb{S}, T(R_1)(r_1) \wedge T(R_2)(r_2) \wedge r = r_1 {+} r_2.$$

For ρ a valuation for R_1, R_2, we have $r_1 = [\![R_1]\!]_\rho$ and $r_2 = [\![R_2]\!]_\rho$. Then the only r that satisfies this predicate is

$$r = r_1 + r_2 = [\![R_1]\!]_\rho + [\![R_2]\!]_\rho = [\![R_1 + R_2]\!]_\rho,$$

as desired.

For a more involved example, consider the translation of $\mathbf{max}(R_1, R_2)$:

$$\begin{aligned}&T(\mathbf{max}(R_1, R_2))(r) \\ &:= \exists r_1\, r_2 : \mathbb{S}, T(R_1)(r_1) \wedge T(R_2)(r_2) \wedge \\ &(r_1 \geq r_2 \wedge r = r_1 \vee r_2 \geq r_1 \wedge r = r_2).\end{aligned}$$

Again, for any valuation ρ of R_1, R_2, we have $r_1 = [\![R_1]\!]_\rho$ and $r_2 = [\![R_2]\!]_\rho$. The final conjunction states that r must be the larger of r_1 and r_2, which is precisely the semantics we have given $[\![\mathbf{max}(R_1, R_2)]\!]_\rho$. The full translation is in Figure 5.

We formalize our intuitive explanation of the translation with the following lemma.

$$
\begin{aligned}
\kappa &:= \mathbb{N} \mid \mathbb{S} \\
T(i)(r) &:= i = r \\
T(R_1 + R_2)(r) &:= \exists r_1\, r_2 : \mathbb{S}, T(R_1)(r_1) \wedge T(R_2)(r_2) \wedge r = r_1 + r_2 \\
T(R_1 \cdot R_2)(r) &:= \exists r_1\, r_2 : \mathbb{S}, T(R_1)(r_1) \wedge T(R_2)(r_2) \wedge r = r_1 \cdot r_2 \\
T(\mathbf{max}(R_1, R_2))(r) &:= \exists r_1\, r_2 : \mathbb{S}, T(R_1)(r_1) \wedge T(R_2)(r_2) \wedge (r_1 \geq r_2 \wedge r = r_1 \vee r_2 \geq r_1 \wedge r = r_2) \\
T(\mathbf{case}(S, R_0, i, R_s))(r) &:= \exists r_s : \mathbb{N}, T(S)(r_s) \wedge (r_s = 0 \wedge T(R_0)(r) \vee \exists i : \mathbb{N}, r_s = i + 1 \wedge T(R_s)(r)) \\
T(\mathbf{sup}(i : \kappa, R))(r) &:= \mathbf{bound}(i : \kappa, R, r) \wedge \forall r', \mathbf{bound}(i : \kappa, R, r') \Rightarrow r' \geq r \\
\mathbf{bound}(i : \kappa, R, r) &:= \forall i : \kappa, \exists r' : \mathbb{S}, T(R)(r') \wedge r' \leq r
\end{aligned}
$$

Figure 5. Constraint Translation

Lemma 7. *For every sensitivity expression R and $r \in \mathbb{S}$, and for every valuation ρ whose domain contains the free variables of R, $[\![T(R)(r)]\!]_\rho \iff r = [\![R]\!]_\rho$*

Proof. By induction on R. We have already considered the $R_1 + R_2$ and $\mathbf{max}(R_1, R_2)$ cases above. □

Using the translation of terms, we can translate sensitivity constraints generated by our typing algorithm. We map each constraint of the form

$$\models \forall \phi, \Phi \Rightarrow R_1 \geq R_2$$

for R_1 a standard sensitivity term to

$$\forall \phi, \Phi \Rightarrow \exists r : \mathbb{S}, T(R_2)(r) \wedge R_1 \geq r$$

Note that since R_1 is a standard sensitivity term, the resulting formula is a first-order formula in the theory of $\mathbb{S}$ and $\mathbb{N}$. Thanks to Lemma 7, both formulas are semantically equivalent.

7. Undecidability of Type-checking over Mixed Reals/Naturals

As we have seen in the previous section, constraints over our extended sensitivity language can be translated to simple first-order formulas. Taken by itself, this is not entirely satisfactory, as the first-order theory of $\mathbb{N}$ is already undecidable. A nice illustration of this is Hilbert's tenth problem, which asks if a polynomial equation of the form $P(\vec{x}) = 0$ over several variables has any solutions over the natural numbers. After several years of investigation, this property, easily definable in first-order arithmetic, was finally shown to be undecidable.

In this section, we will show that this problem is present in *DFuzz*: type-checking is undecidable. We begin with an auxiliary lemma.

Lemma 8. *Given polynomials P, Q over n variables with coefficients in $\mathbb{N}$, checking $\forall \vec{i} \in \mathbb{N}^n, P(\vec{i}) \geq Q(\vec{i})$ is undecidable.*

Proof. We will use a solution to our problem to solve Hilbert's tenth problem. Suppose we are given a polynomial P with integer coefficients, and we want to decide whether $\exists \vec{i} \in \mathbb{N}^n, P(\vec{i}) = 0$. This is equivalent to deciding $\neg \forall \vec{i} \in \mathbb{N}^n, P(\vec{i})^2 \geq 1$. Write $P(\vec{i})^2 = P^+(\vec{i}) - P^-(\vec{i})$, where P^+ and P^- have only positive coefficients. Then our condition is equivalent to $\neg \forall \vec{i} \in \mathbb{N}^n, P^+(\vec{i}) \geq P^-(\vec{i}) + 1$. Thus, we can solve Hilbert's tenth problem by using P^+ and $P^- + 1$ as inputs to our problem, which shows that it is undecidable. □

This class of constraints is important for *DFuzz*, as they can arise when checking the subtype relation.

Corollary 9. *The subtype relation of* DFuzz *is undecidable.*

Proof. Suppose we are given P and Q as previously. Consider the types $\sigma = \forall \vec{i}, !_0 \mathbb{N}^n[\vec{i}] \multimap !_{Q(\vec{i})} \mathbb{R} \multimap \mathbb{R}$ and $\tau = \forall \vec{i}, !_0 \mathbb{N}^n[\vec{i}] \multimap !_{P(\vec{i})} \mathbb{R} \multimap \mathbb{R}$. Then $\sigma \sqsubseteq \tau$ is equivalent to the previous problem, hence undecidable. □

Corollary 10. DFuzz *type checking is undecidable.*

Proof. Using recursion and dependent pattern matching, it is possible to write a function that multiplies a real number by a polynomial $Q(\vec{v})$ with variables ranging over $\mathbb{N}$. Its minimal type will clearly be σ. Therefore, type-checking it against τ is equivalent to deciding $\sigma \sqsubseteq \tau$, which is undecidable by Lemma 8. □

8. Approaches to Constraint Solving

Given that type-checking *DFuzz* (and hence also *EDFuzz*) is undecidable, is there anything more we can do besides feeding the constraints to a solver and hoping for the best? In this section, we discuss two possible directions to tackle these constraints. For both of these approaches, we require that *all annotations in the term are standard sensitivities*, rather than extended. Then, we have the following lemma. (We defer the proof to the Appendix.)

Lemma 11 (Standard Annotations)**.** *Assume annotations in a term e range over standard sensitivities and $\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Gamma; \sigma$. Then:*

- *σ has no extended sensitivities; and*
- *all constraints required for the algorithm are of the form $\models \forall \phi.\ (\Phi \Rightarrow R \geq R')$ where R is a standard sensitivity term.*

8.1 Modifying the subtype relation

As seen in the previous section, the *EDFuzz* subtyping relation is undecidable. Here, we explore a modified version of *EDFuzz*—which we call *UDFuzz*—that enjoys decidable typechecking. The modification is simple to describe: *UDFuzz* has all the same typing rules as *EDFuzz*, except we strongly restrict the subtyping relation to force all generated constraints to be decidable, and all annotations must be standard sensitivity terms. By restricting the subtype relation of *EDFuzz*, *UDFuzz* typeable programs are a strict subset of *EDFuzz*. This subtype restriction will rule out many programs that are typeable under *EDFuzz*, but is expressive enough to cover a range of examples (including most of the examples presented in the original work on *DFuzz* [10]).

Recall that the constraints handled by our algorithmic system have the form

$$\models \forall \phi.\ \Phi \Rightarrow R \geq R',$$

where R, R' are possibly extended sensitivity terms, and ϕ consists of both natural and real index variables. As we are requiring all annotations in *UDFuzz* standard sensitivities, then by Lemma 11, R will be a standard sensitivity term in *UDFuzz*; we use this invariant to show the subtype relation of *UDFuzz* is a subrelation of the subtype relation of *EDFuzz*.

Furthermore, we note that the first order theory over $\mathbb{S}$ is decidable: we can try all settings variables to ∞ and check the resulting constraints (with all the remaining quantifiers ranging over $\mathbb{R}$). The resulting formula is in the first order theory over $\mathbb{R}$, and is decidable (as shown by Tarski). Hence, a natural idea is to replace quantification over the naturals with quantification over $\mathbb{S}$; let us first make this idea precise.

We define the semantics for sensitivity terms, where natural-kinded free variables may now be mapped to values in $\mathbb{S}$. We call this extension the *uniform* interpretation of size and sensitivity terms, and denote it by $[\![\cdot]\!]^U$. A well-formed *uniform valuation* $\phi \models^U \rho$ maps $\mathrm{dom}(\phi)$ to $\mathbb{S}$; note that "size variables" may be interpreted as real numbers, not just natural numbers.

First, the uniform interpretation of standard size and sensitivity terms is completely identical to the standard interpretation. The extended sensitivities have slightly different interpretations: $\mathbf{sup}(i, R)$ now takes a max over all real numbers, and $\mathbf{case}(S, R_0, i, R_s)$ must now be defined when the interpretation of S is not an integer.

$$
\begin{aligned}
[\![\mathbf{sup}(i : \kappa, \hat{R})]\!]^U_\rho &:= \sup_{r \in \mathbb{S}} \{[\![\hat{R}]\!]^U_{\rho \cup [i=r]}\} \\
[\![\mathbf{max}(\hat{R}_1, \hat{R}_2)]\!]^U_\rho &:= \max([\![\hat{R}_1]\!]^U_\rho, [\![\hat{R}_2]\!]^U_\rho) \\
[\![\mathbf{case}(S, \hat{R}_0, i, \hat{R}_s)]\!]^U_\rho &:= \begin{cases} [\![\hat{R}_1]\!]^U_\rho & \text{if} \quad [\![S]\!]^U_\rho = 0 \\ 0 & \text{if} \quad [\![S]\!]^U_\rho \in (0,1) \\ [\![\hat{R}_2]\!]^U_{\rho \cup [i=r-1]} & \text{if} \quad [\![S]\!]^U_\rho = r \geq 1. \end{cases} \\
[\![\hat{R}_1 + \hat{R}_2]\!]^U_\rho &:= [\![\hat{R}_1]\!]_\rho + [\![\hat{R}_2]\!]^U_\rho \\
[\![\hat{R}_1 \cdot \hat{R}_2]\!]^U_\rho &:= [\![\hat{R}_1]\!]^U_\rho \cdot [\![\hat{R}_2]\!]^U_\rho \\
[\![R]\!]^U_\rho &:= [\![R]\!]^U_\rho \qquad \text{otherwise.}
\end{aligned}
$$

We first show that this uniform semantics is an extension of the standard semantics.

Lemma 12. *Suppose R is a standard sensitivity term, typed under context ϕ. Then, for any standard valuation $\phi \models \rho$, we have*

$$[\![R]\!]^U_\rho = [\![R]\!]_\rho.$$

Proof. Immediate from the definition of the interpretation. □

Now, we can define the uniform interpretation of constraints. A constraint

$$\models^U \forall \phi.\, \Phi \Rightarrow R \geq R'$$

is true exactly when for all real-valued valuations $\phi \models^U \rho$ satisfying Φ, we have $[\![R]\!]^U_\rho \geq [\![R']\!]^U_\rho$.

We are now ready to prove that the uniform interpretation of constraints is sound with respect to the original interpretation.

Theorem 13. *Suppose R, R' are well-typed in context ϕ. Suppose that*

$$\models^U \forall \phi.\, \Phi \Rightarrow R \geq R',$$

for R a standard sensitivity term. Then,

$$\models^U \forall \phi.\, \Phi \Rightarrow R \geq R'.$$

Proof. It suffices to show that for any standard valuation $\phi \models \rho$, we have $[\![R']\!]^U_\rho \geq [\![R']\!]_\rho$. (We defer the proof of this claim to the long version.) Assuming this, the theorem assumption shows that for all standard valuation $\phi \models \rho$, we have

$$[\![R]\!]^U_\rho \geq [\![R']\!]^U_\rho \geq [\![R']\!]_\rho.$$

But R is a standard sensitivity, so $[\![R]\!]^U_\rho = [\![R]\!]_\rho$ by Lemma 12, and we are done. □

Hence, the subtype relation of *UDFuzz* is a subrelation of the subtype relation in *EDFuzz*. By reasoning analgous to Lemma 7, we can show that relaxing the first order translation of constraints captures this uniform interpretation. More formally:

Lemma 14. *For every sensitivity term R, let $T^U(R)$ be a unary predicate defined exactly as in Figure 5, but replacing quantification over $\mathbb{N}$ with quantificiation over $\mathbb{S}$ and with the modified* **case** *translation:*

$$
\begin{aligned}
&T^U(\mathbf{case}(S, R_0, i, R_s))(r) := \\
&\quad \exists r_s : \mathbb{N}, \quad T(S)(r_s) \wedge (r_s = 0 \wedge T(R_0)(r)) \\
&\quad\quad \vee \quad (0 < r_s < 1 \wedge r = 0) \\
&\quad\quad \vee \quad (\exists i : \mathbb{N}, r_s = i + 1 \wedge T(R_s)(r))
\end{aligned}
$$

Then, $r \in \mathbb{S}$, and for every uniform valuation ρ whose domain contains the free variables of R, $[\![T^U(R)(r)]\!]^U_\rho \iff r = [\![R]\!]^U_\rho$.

By this lemma, we can give a sound, complete and decidable type-checking algorithm for *UDFuzz*.

Theorem 15. *Suppose we use our algorithmic system, with the constraints*

$$\models^U \forall \phi.\, \Phi \Rightarrow R_1 \geq R_2$$

handled by translation to the first order formula

$$\forall \phi, \Phi \Rightarrow \exists r : \mathbb{S}, T^U(R_2)(r) \wedge R_1 \geq r,$$

where all quantifiers are over $\mathbb{S}$. Since the theory of $\mathbb{S}$ is decidable, this gives an effective type-checking procedure for UDFuzz*.*

Proof. Note that R_1 is a standard sensitivity term, so the translated formula is indeed a first order formula over the theory of $\mathbb{S}$. By Lemma 14, the translated formula is logically equivalent to

$$[\![\Phi]\!]^U_\rho \Rightarrow [\![R_1]\!]^U_\rho \geq [\![R_2]\!]^U_\rho$$

for all uniform valuations $\phi \models^U \rho$, which in turn implies $\phi; \Phi \models R_1 \geq R_2$ by Theorem 13. This shows that the algorithmic system is sound and complete with respect to *UDFuzz*. □

Remark 16. UDFuzz *is a strict subset of* EDFuzz*; informally, it contains* EDFuzz *programs with typing derivations that do not use facts true over $\mathbb{N}$ but not over $\mathbb{R}$. One key way that subtyping is used in* EDFuzz *is for equational manipulations of the indices; for instance, subtyping may be needed to change the index expression $3(i+1)$ to $3i+3$. This reasoning is available in* UDFuzz *as well; indeed, most of the example programs in* DFuzz *are typeable under* UDFuzz *as well. (The only exception is k-medians, which extends the index language with a division function that we do not handle.)*

However, there are many programs that lie in EDFuzz *but not in* UDFuzz*—constraints as simple as $\forall i.\ i^2 \geq i$ are true when quantifing over the naturals but not when quantifying over the reals. Valid* EDFuzz *programs that use these facts in their typing derivation will not lie in* UDFuzz*.*

8.2 Constraint Simplification

Rather than restricting the subtype relation, we can also try to generate simpler constraints when type-checking *EDFuzz*. While the translation of extended constraints to first order real theory is conceptually simple, the translation generates complex constraints; in particular, they may have many alternating quantifiers. In this section, present a rewriting procedure for reducing extended sensitivity terms, leading to simpler constraints. We continue to require that all source annotations must be standard sensitivity terms.

To begin, we generalize our three extended constructs with a new *constrained least upper bound* (**club**) operation, with form $\mathbf{club}\{(\phi_1; \Phi_1; R_1), \ldots, (\phi_n; \Phi_n; R_n)\}$. Here, ϕ is a size and sensitivity variable context, Φ is a constraint context, and R is a sensitivity term, extended or standard. The judgment for a well-formed

club is

$$\phi \models \mathbf{club}\{(\phi_1; \Phi_1; R_1), \dots, (\phi_n; \Phi_n; R_n)\},$$

where each R_j has kind r under $\phi, \phi_j; \Phi_j$, and $\phi, \{\phi_j\}_j$ have disjoint domain. Intuitively, **club** is a maximum over a set of sensitivities, restricting to sensitivities where the associated constraint is satisfied. Sensitivities where the constraints are not satisfied are ignored. Formally, let ϕ contain the free variables of **club**, and let $\phi \models \rho$ be any standard valuation. We can give the following interpretation of **club**:

$$(\!|\mathbf{club}\{(\phi_1; \Phi_1; R_1), \dots, (\phi_n; \Phi_n; R_n)\}|\!)_\rho :=$$
$$\max_{j \in [n]} \max\{[\![R_j]\!]_{\rho \cup \rho_j} \mid \phi_j \models \rho_j \text{ and } \rho, \rho_j \models \Phi_j\}.$$

We define the maximum over an empty set to be 0.

Now, we can encode the extended sensitivity terms using only **club**, through the following translation function:

$$\begin{aligned}
C(\mathbf{max}(\hat{R}_1, \hat{R}_2)) &:= \mathbf{club}\{(\emptyset; \emptyset; C(\hat{R}_1)), (\emptyset; \emptyset; C(\hat{R}_2))\} \\
C(\mathbf{sup}(i, \hat{R})) &:= \mathbf{club}\{(i; \emptyset; C(\hat{R}))\} \\
C(\mathbf{case}(S, i, \hat{R}_0, \hat{R}_s)) &:= \mathbf{club}\{(\emptyset; S = 0; C(\hat{R}_0)), \\
&\qquad (i; S = i + 1; C(\hat{R}_s))\} \\
C(\hat{R}_1 + \hat{R}_2) &:= C(\hat{R}_1) + C(\hat{R}_2) \\
C(\hat{R}_1 \cdot \hat{R}_2) &:= C(\hat{R}_1) \cdot C(\hat{R}_2) \\
C(R) &:= R \qquad \text{otherwise.}
\end{aligned}$$

While we may now have nested **club**, we extend the interpretation in the natural way. We can show that the translation faithfully preserves the semantics of the extended terms, with the following lemma.

Lemma 17. *Suppose $\phi \models R$ and $\phi \models \rho$ is a standard valuation. Then, $(\!|C(R)|\!)_\rho = [\![R]\!]_\rho$.*

Proof. By induction on R. □

Now, we can simplify the compiled constraints. First, we can push all standard sensitivity terms to the leaves of the expression. More formally, we have the following lemma.

Lemma 18. *Suppose $\phi \models R \cdot \mathbf{club}\{(\phi_i; \Phi_i; C_i)\}_i + R'$, where R, R' are standard sensitivity terms, and C_i is an arbitrary sensitivity term possibly involving* **club**. *Then, for any standard closing valuation $\phi \models \rho$,*

$$(\!|R \cdot \mathbf{club}\{(\phi_i; \Phi_i; C_i)\}_i + R'|\!)_\rho = (\!|\mathbf{club}\{(\phi_i; \Phi_i; R \cdot C_i + R')\}_i|\!)_\rho.$$

Proof. By the definition of the interpretations, and the mathematical fact

$$a \cdot \max_i\{b_i\} + c = \max_i\{a \cdot b_i + c\},$$

for $a, b, c \geq 0$. □

Thus, without loss of generality we may reduce the compiled sensitivity constraint to an expression of the form Q, with grammar

$$Q ::= \emptyset \mid Q_1 + Q_2 \mid Q_1 \cdot Q_2 \mid \mathbf{club}\{(\phi_i; \Phi_i; Q_i)\} \mid \mathbf{club}\{(\phi_i; \Phi_i; R_i)\},$$

where R_i are standard sensitivity terms. We will use the metavariable V to denote an arbitrary (possibly empty) collection of triples $(\phi_i; \Phi_i; R_i)_i$, and the metavariable W to denote an arbitrary (possibly empty) collection of triples $(\phi_i; \Phi_i; Q_i)_i$. Throughout, we will implicitly work up to permutation of the arguments to **club**: for instance, $\mathbf{club}\{(X), (Y)\}$ will be considered the same as $\mathbf{club}\{(Y), (X)\}$. We will also work up to commutativity of addition and multiplication: $Q_1 + Q_2$ will be considered the same as $Q_2 + Q_1$, and likewise with multiplication. We present the constraint simplification rules as a rewrite relation $\mapsto$. As typical, we will write $\mapsto^*$ for the reflexive, transitive closure of $\mapsto$. The full rules are in Figure 6.

We can prove correctness of our constraint simplification with the following lemma.

Lemma 19. *Suppose $Q \mapsto Q'$, and suppose $\phi \models Q$ and $\phi \models Q'$. Then, for any standard valuation $\phi \models \rho$, we have $(\!|Q|\!)_\rho = (\!|Q'|\!)_\rho$.*

Proof. By induction on the derivation of $Q \mapsto Q'$. The cases Plus, Mult and Red are immediate by induction. The other cases all follow by the semantics of **club**; details are in the Appendix. □

The simplification relation terminates in the following particular simple form.

Lemma 20. *Let Q be a sensitivity term involving* **club**. *Along any reduction path, Q reduces in finitely many steps to a term of the form*

$$\mathbf{club}\{V\} = \mathbf{club}\{(\phi_1; \Phi_1; R_1), \dots, (\phi_n; \Phi_n; R_n)\}.$$

Proof. First, note that any reduction of Q must terminate in finitely many steps: by induction on the derivation of the reduction, it's clear that each reduction removes one **club** subterm, and no reductions introduce **club** subterms. So, suppose that Q is a term with no possible reductions.

By induction on the structure of Q, we claim that Q is of the desired form. Say if $Q = Q_1 + Q_2$, if either Q_1, Q_2 can reduce, then Plus applies. If not, then by induction, CPlus applies. The same reasoning follows for $Q = Q_1 \cdot Q_2$: either Mult applies, or CMult does. Finally, if Q is a single **club** term, if Red and Flat both don't apply, then Q is of the desired form. □

Finally, checking a constraint $\forall \phi.\, \Phi \Rightarrow R \geq \mathbf{club}\{V\}$ is simple.

Lemma 21. *Let R be a standard sensitivity term, and let V be*

$$V = (\phi_1; \Phi_1; R_1), \dots, (\phi_n; \Phi_n; R_n)$$

where each R_j is a standard sensitivity term without **club**. *Then, $\models \forall \phi.\, \Phi \Rightarrow R \geq \mathbf{club}\{V\}$ is logically equivalent to*

$$\forall_{j \in [n]} \phi, \phi_j.\, \Phi \Rightarrow \bigwedge_{k \in [n]} (\Phi_k \Rightarrow R \geq R_k).$$

Proof. Immediate by the semantics of $\mathbf{club}\{V\}$. □

Putting together all the pieces, for a constraint

$$\models \forall \phi.\, \Phi \Rightarrow R \geq R',$$

with R standard, we can transform $C(R')$ to a term of the form Q by pushing all standard sensitivity terms to the leaves. Then, we normalize $Q \mapsto^* \mathbf{club}\{V\}$ by Lemma 20 arbitrarily. By Lemma 19, the interpretation of Q and $\mathbf{club}\{V\}$ are the same, so we can reduce the constraint $\models \forall \phi.\ \phi \Rightarrow R \geq \mathbf{club}\{V\}$ to a first order formula over mixed naturals and $\mathbb{S}$, with no alternating quantifiers, by Lemma 21.

9. Related work

There is a vast literature on type checking for various combinations of indexed types, linear types, dependent types and subtyping. A distinctive feature of our approach is that our index language represents natural and real number expressions. As we have shown in the previous sections, this makes type checking non-trivial.

The work most closely related to ours is Dal Lago et al. [6], who studied the type inference problem for dℓPCF, a relatively-complete type system for complexity analysis introduced in Dal Lago and Gaboardi [4]. dℓPCF uses ideas similar to *DFuzz* but brings the idea

$$\frac{}{\mathbf{club}\{(\phi;\Phi;\mathbf{club}\{(\phi_i;\Phi_i;R_i)\}_i),V\}\mapsto\mathbf{club}\{(\phi\cup\phi_i;\Phi\wedge\Phi_i;R_i),V\}_i}\ \text{Flat}$$

$$\frac{}{\mathbf{club}\{(\phi_i;\Phi_i;R_i)\}_i+\mathbf{club}\{(\phi'_j;\Phi'_j;R'_j)\}_j\mapsto\mathbf{club}\{(\phi_i\cup\phi'_j;\Phi_i\wedge\Phi'_j;R_i+R'_j)\}_{ij}}\ \text{CPlus}$$

$$\frac{}{\mathbf{club}\{(\phi_i;\Phi_i;R_i)\}_i\cdot\mathbf{club}\{(\phi'_j;\Phi'_j;R'_j)\}_j\mapsto\mathbf{club}\{(\phi_i\cup\phi'_j;\Phi_i\wedge\Phi'_j;R_i\cdot R'_j)\}_{ij}}\ \text{CMult}$$

$$\frac{Q_1\mapsto Q'_1}{Q_1+Q_2\mapsto Q'_1+Q'_2}\ \text{Plus}\qquad\frac{Q_1\mapsto Q'_1}{Q_1\cdot Q_2\mapsto Q'_1\cdot Q_2}\ \text{Mult}\qquad\frac{Q\mapsto Q'}{\mathbf{club}\{(\phi;\Phi;Q),W\}\mapsto\mathbf{club}\{(\phi;\Phi;Q'),W\}}\ \text{Red}$$

Figure 6. $\mathbf{club}$ Reduction

of linear dependent types to the limit. Indeed, dℓPCF index language contains function symbols that are given meaning by an equational program. The equational program then plays the role of an oracle for the type system—dℓPCF is in fact a family of type systems parametrized over the equational program. The main contribution of Dal Lago et al. [6] is an algorithm that, given a PCF program, generates a type and the set of constraints that must be satisfied in order to assign the return type to the input term.

In our terminology, their work is similar to the top-down approach we detailed in Section 3. As we discussed there, the complication of this approach is that it requires solving constraints over expressions—with possible function symbols—of the index-level language. As shown by Dal Lago and Petit, a clear advantage of the dℓPCF formulation is that instead of introducing an existential variable over expressions, one can introduce a new function symbol that will then be given meaning by the equational program generated by the constraints—i.e., the constraints give a description of the semantics of the program, which can be turned in an equational program, that in turn gives meaning to the function symbols of the index language appearing in the type. Clearly, this approach cannot be reduced to numeric resolution and need instead a combination of numeric and symbolic solving technology. The authors show that these constraints can be anyway handled by using the WHY3 framework. Some constraints are discharged automatically by some of the solvers available in WHY3 while others requires an interactive resolution using Coq.

As explained in Section 3, the situation with *DFuzz* is different. Indeed, *DFuzz* can be seen as a simplified version of dℓPCF—simplifying in particular the typing for the fixpoint and without variable bindings in !-types—extended however to deal with indices representing real numbers and using quantifications over index variables. A key distinction of *DFuzz* is that the set of constructors for the language of sensitivity is *fixed*—one cannot add arbitrary functions. Moreover, the extension to real numbers gives a different behavior from how natural numbers are used in dℓPCF—e.g., our example for the lack of minimal type would make no sense in dℓPCF. These distinctions make the type checking problem very different.

For another approach that is closely related to our work, recall that *DFuzz* is an extension of *Fuzz*. The sensitivity inference and sensitivity checking problems for *Fuzz* have been studied in D'Antoni et al. [7]. These problems are simpler than the one studied here since in *Fuzz* there is no dependency, no quantification and no subtyping. Indeed, the constraints generated are much simpler and can be solved quickly by an SMT solver.

Similarly, Eigner and Maffei [9] have studied an extension of *Fuzz* for modeling protocols. In their work they also give an algorithmic version of their type system. Their type system presents challenges similar to *Fuzz*, which they handle with algebraic manipulations. More precisely, their algorithmic version uses a technique similar to the one developed in Cervesato et al. [2] for the splitting of resources: when a rule with multiple premises is encountered the algorithmic system, first allocate all the resources to the first branch and then allocate the remaining resources to the second branch. Unfortunately, this approach cannot be easily applied to *DFuzz* due to the presence of index variables and dependent pattern matching.

From a different direction, recent works [1, 13] have shown how linear indexed type systems can be made more abstract and useful to analyze abstract resources. In particular, this kind of analyses is connected to comonadic notions of computations [20]. The type inference algorithm described in Ghica and Smith [13] is parametric on an abstract notion of resource. This resource can be instantiated on a language for sensitivities similar to the one in *Fuzz*. So, this abstract type inference procedure could be also used for sensitivity analysis.

DFuzz is one of several languages combining linear and dependent types. For example, ATS [3] is designed around a dependent type system enriched with a notion of resources that is a type-level representation of memory locations; these resources are managed using a linear discipline. ATS uses these features to verify the correctness of memory and pointer management.

Even if the use of linear types in ATS is very different from the one presented here, our type checking algorithm shares some similarities with ATS's one. The main difference is that ATS uses interactive theorem proving to discharge proof obligations while, thanks to the restricted scope of our analysis, our constraints can be handled by numeric solvers. In contrast, DML [26]—a predecessor of ATS which did not use linear types—uses an approach similar to ours by solving proof obligations using automatic numeric resolution. This required limitations on the operations available in the index language, similar to *DFuzz*.

Another work considering lightweight dependent types is the one by Zhu and Jagannathan [27]. In particular they propose a technique based on dependent types to reduce the verification of higher order programs to the verification of a first order language. While the goal of their work is similar in spirit to ours, their technique has only superficial similarities with the one presented here.

Finally, our work has been informed by the wide literature on type-checking, far too large to summarize here. For instance, the problem of dealing with subtyping rules by using syntax-directed systems has been studied by Pierce and Steffen [21], and others.

10. Conclusions and Future Work

We have presented a type-checking and inference algorithm for *EDFuzz*—a simple extension of *DFuzz*—featuring a linear indexed dependent type system. While we have shown that *DFuzz* type-checking is undecidable in the general case, our approach generates constraints over the first order theory over the reals and naturals, for which there are standard (though necessarily incomplete) solvers.

We are currently experimenting with a prototype implementation;[3] more investigation is needed in order to assess the difficulty of these constraints on real examples.

Overall, our design was guided by two principles: to stay as close to *DFuzz* as possible, and to provide a practical type checking procedure. While we do require extensions to *DFuzz*, there is a clear motivation for the introduction of each new construct. The idea of making a limited enrichment of the index language in order to simplify type-checking may be applicable to other linear indexed type systems. Furthermore, designers of such systems would do well to keep implementability in mind: seemingly unimportant decisions that simplify the metatheory may have a serious impact on type-checking.

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$$\frac{\phi; \Phi \mid \Delta \vdash e : \sigma \qquad \phi; \Phi \models \Gamma \sqsubseteq \Delta}{\phi; \Phi \mid \Gamma \vdash e : \sigma} \ (\sqsubseteq.\mathrm{L}) \qquad \frac{\phi; \Phi \mid \Gamma \vdash e : \sigma \qquad \phi; \Phi \models \sigma \sqsubseteq \tau}{\phi; \Phi \mid \Gamma \vdash e : \tau} \ (\sqsubseteq.\mathrm{R}) \qquad \frac{}{\phi; \Phi \mid \Gamma \vdash \mathrm{r} : \mathbb{R}} \ (\mathrm{Const}_{\mathbb{R}})$$

$$\frac{n = [\![S]\!]}{\phi; \Phi \mid \Gamma \vdash n : \mathbb{N}[S]} \ (\mathrm{Const}_{\mathbb{N}}) \qquad \frac{}{\phi; \Phi \mid \Gamma, x :_{[1]} \sigma \vdash x : \sigma} \ (\mathrm{Var}) \qquad \frac{\phi; \Phi \mid \Gamma_1 \vdash e_1 : \sigma \qquad \phi; \Phi \mid \Gamma_2 \vdash e_2 : \tau}{\phi; \Phi \mid \Gamma_1 + \Gamma_2 \vdash (e_1, e_2) : \sigma \otimes \tau} \ (\otimes I)$$

$$\frac{\phi; \Phi \mid \Delta \vdash e : \sigma \otimes \tau \qquad \phi; \Phi \mid \Gamma, x :_{[R]} \sigma, y :_{[R]} \tau \vdash e' : \mu \qquad R \neq \square}{\phi; \Phi \mid \Gamma + R \cdot \Delta \vdash \mathbf{let}(x, y) = e \ \mathbf{in}\ e' : \mu} \ (\otimes E) \qquad \frac{\phi; \Phi \mid \Gamma \vdash e_1 : \sigma \qquad \phi; \Phi \mid \Gamma \vdash e_2 : \tau}{\phi; \Phi \mid \Gamma \vdash \langle e_1, e_2 \rangle : \sigma \ \& \ \tau} \ (\& \ I)$$

$$\frac{\phi; \Phi \mid \Gamma \vdash e : \sigma_1 \ \& \ \sigma_2}{\phi; \Phi \mid \Gamma \vdash \pi_i \ e : \sigma_i} \ (\& \ E) \qquad \frac{\phi; \Phi \mid \Gamma, x :_{[R]} \sigma \vdash e : \tau \qquad R \neq \square}{\phi; \Phi \mid \Gamma \vdash \lambda(x :_{[R]} \sigma).e : !_R \sigma \multimap \tau} \ (\multimap I)$$

$$\frac{\phi; \Phi \mid \Gamma \vdash e_1 : !_R \sigma \multimap \tau \qquad \phi; \Phi \mid \Delta \vdash e_2 : \sigma}{\phi; \Phi \mid \Gamma + R \cdot \Delta \vdash e_1 \ e_2 : \tau} \ (\multimap E) \qquad \frac{\phi, i : \kappa; \Phi \mid \Gamma \vdash e : \sigma \qquad i \text{ fresh in } \Phi, \Gamma}{\phi; \Phi \mid \Gamma \vdash \Lambda i : \kappa.\ e : \forall i : \kappa.\ \sigma} \ (\forall I)$$

$$\frac{\phi; \Phi \mid \Gamma \vdash e : \forall i : \kappa.\ \sigma \qquad \phi \models S : \kappa}{\phi; \Phi \mid \Gamma \vdash e[S] : \sigma[S/i]} \ (\forall E) \qquad \frac{\phi; \Phi \mid \Gamma, x :_{[\infty]} \sigma \vdash e : \sigma}{\phi; \Phi \mid \infty \cdot \Gamma \vdash \mathbf{fix}\, x : \sigma.\ e : \sigma} \ (\mathrm{Fix}) \qquad \frac{\phi; \Phi \mid \Gamma \vdash e : \mathbb{N}[S]}{\phi; \Phi \mid \Gamma \vdash e + 1 : \mathbb{N}[S + 1]} \ (\mathrm{S}\ I)$$

$$\frac{\begin{array}{c}\phi; \Phi \mid \Delta \vdash e : \mathbb{N}[S] \qquad \phi; \Phi, S = 0 \mid \Gamma \vdash e_0 : \sigma \\ \phi, i : \mathrm{n}; \Phi, S = i + 1 \mid \Gamma, n :_{[R]} \mathbb{N}[i] \vdash e_s : \sigma \qquad i \# R \qquad R \neq \square\end{array}}{\phi; \Phi \mid \Gamma + R \cdot \Delta \vdash \mathbf{case}\, e\, \mathbf{return}\, \sigma\, \mathbf{of}\ 0 \Rightarrow e_0 \mid n_{[i]} + 1 \Rightarrow e_s : \sigma} \ (\mathbb{N}\ E)$$

Figure 7. *DFuzz* $_\square$ Type Judgment

A. Differences Compared to Gaboardi et al. [10]

While we hew closely to the presentation of *DFuzz* in Gaboardi et al. [10], we make a few technical changes.

- The context weakening operation $\Gamma \sqsubseteq \Gamma'$ in *DFuzz* allows the types to change. That is, a binding $x :_{[R]} \sigma \in \Gamma$ can be weakened to $x :_{[R']} \sigma'$ for $\sigma \sqsubseteq \sigma'$ two syntactically different types. We take a more restricted weakening rule, where the types must be syntactically the same; we are unaware of any programs that need the more general rule.
- We take the interpretation of $\infty \cdot 0$ to be ∞, rather than 0.
- We assume some additional type annotations in the source language, as discussed in Section 5

B. The *DFuzz* $_\square$ system

The first system has the goal to enjoy "context" uniformity, in the sense that sensitivity information in the contexts may be missing. We denote such an assignment $x :_\square \sigma$. This is a subtle technical point for crucial to enable syntax-directed typability.

We modify subtyping for environments such that $\Gamma \sqsubseteq \Delta$ requires Γ, Δ to have the same domain. The new rule is:

$$\frac{\begin{array}{c}\forall (x_i :_{[R_i]} \sigma_i, x_i :_{[R'_i]} \sigma_i) \in (\Gamma, \Delta) \\ \mathrm{dom}(\Delta) = \mathrm{dom}(\Gamma) \qquad \models \forall \phi.\ (\Phi \Rightarrow R_i \geq R'_i) \vee R'_i = \square\end{array}}{\phi; \Phi \models \Gamma \sqsubseteq \Delta} \ \sqsubseteq\text{-Env}$$

This subsumes regular variable weakening. Context operations must be aware of $\square$, with $\square + i = i$, $i \cdot \square = \square$ for the annotations.

Definition 22 (Box erasure)**.** *For any context Γ, we define the $\square$-erasure operation $|\Gamma| = \{x :_{[R]} \sigma \mid x :_{[R]} \sigma \in \Gamma \wedge R \neq \square\}$.*

We introduce the $\square$ system in Figure 7.

We prove that derivations in a system with $\square$ are in direct correspondence with derivation in a system without it.

Lemma 23. *Assume $\phi; \Phi \mid \Gamma \vdash e : \sigma$ in the $\square$ system, then $\phi; \Phi \mid |\Gamma| \vdash e : \sigma$ in the system without it.*

Proof. By induction on the typing derivation. The base cases and cases where the context is not modified are immediate. Subtyping on the left is proven by weakening.

The rest of cases are split in two:

- All cases featuring variables in the top rule, also have the condition $R \neq \square$, this is enough.
- For the cases involving context operations, the proofs is completed by following properties:

$$|R \cdot \Gamma| = R \cdot |\Gamma| \qquad |\Gamma + \Delta| = |\Gamma| + |\Delta|$$

□

Lemma 24. *Assume $\phi; \Phi \mid \Gamma \vdash e : \sigma$ in the system without $\square$, then $\phi; \Phi \mid \Gamma \vdash e : \sigma$ in the system with it.*

Proof. The proof is mostly routine by induction on the derivation, but relies in the following fact of the $\square$ system: $\phi; \Phi \mid \Gamma \vdash e : \sigma$ implies $\phi; \Phi \mid \Gamma, x :_{\square} \tau \vdash e : \sigma$. Then, using this lemma we can adjust the contexts so that subtyping goes through in the system with $\square$. $\square$

A $\square$-elimination operation $R_{\square\uparrow}$, which sends context annotations to sensitivities will prove useful in the the syntax directed system. It is defined as $\square_{\square\uparrow} = 0$, $R_{\square\uparrow} = R$ otherwise. Remember that $\square$ doesn't belong to the sensitivity language, so any annotation that is used in places where a sensitivity is expected must be wrapped with $-_{\square\uparrow}$.

Definition 25 (Extension to environments operations)**.** *Operations on extended sensitivites that were extended to environments in a pointwise fashion, now must take into account the presence of* $\square$*.*

- $\mathbf{max}(R_1, R_2)$ *operates now as* $\mathbf{max}(\square, \square) = \square$*,* $\mathbf{max}(\square, R) = R$*,* $\mathbf{max}(R, \square) = R$*, the original term otherwise.*
- $\mathbf{sup}(i, R)$ *is extended in the natural way* $\mathbf{sup}(i, \square) = \square$*, the original term otherwise.*
- $\mathbf{case}(S, i, R_0, R_s)$ *operates now* $\mathbf{case}(S, i, \square, \square) = \square$*,* $\mathbf{case}(S, i, R_0, R_s) = \mathbf{case}(S, i, R_{0\square\uparrow}, R_{s\square\uparrow})$ *otherwise.*

C. Subtyping Proofs

From now on we can consider only contexts of similar length. We prove a few necessary facts about subtyping.

Lemma 26 (Context manipulation)**.** *Context subtyping is preserved by addition and scalar multiplication. More formally:*

- *If* $\phi; \Phi \models \Gamma \sqsubseteq \Gamma' \wedge \Delta \sqsubseteq \Delta'$*, then* $\phi; \Phi \models \Gamma + \Delta \sqsubseteq \Gamma' + \Delta'$*; and*
- *if* $\phi; \Phi \models \Gamma \sqsubseteq \Gamma' \wedge R \geq R'$*, then* $\phi; \Phi \models R \cdot \Gamma \sqsubseteq R' \cdot \Gamma'$*.*

Proof. These follow from the interpretation of subtyping assertions. Note that the subtyping relation preserves the skeleton of the environments, thus making sure that the operations are always defined. $\square$

Lemma 27 (Properties of extended sensitivities)**.** *Extended sensitivities satisfy the following properties:*

- $\phi; \Phi \models R \geq \mathbf{max}(R_1, R_2)$ *if and only if* $\phi; \Phi \models R \geq R_1 \wedge R \geq R_2$*;*
- $\phi; \Phi \models R \geq \mathbf{sup}(i, R')$ *with* $i\#\phi$ *if and only if* $\phi, i; \Phi \models R \geq R'$*; and*
- $\phi; \Phi \models R \geq \mathbf{case}(S, i, R_0, R_s)$ *with* $i\#\phi$ *if and only if*

$$\phi; \Phi, S = 0 \models R \geq R_0 \quad \textit{and} \quad \phi, i; \Phi, S = i + 1 \models R \geq R_s.$$

As an immediate corollary, setting R *to be* $\mathbf{max}(R_1, R_2), \mathbf{sup}(i, R'), \mathbf{case}(S, i, R_0, R_s)$ *yields*

- $\phi; \Phi \models \mathbf{max}(R_1, R_2) \geq R_1 \wedge R \geq R_2$*;*
- $\phi, i; \Phi \models \mathbf{sup}(i, R') \geq R'$*; and*
- $\phi; \Phi, S = 0 \models \mathbf{case}(S, i, R_0, R_s) \geq R_0$ *and* $\phi, i; \Phi, S = i + 1 \models \mathbf{case}(S, i, R_0, R_s) \geq R_s$*.*

Proof. These follow from the interpretation of extended sensitivities. $\square$

Lemma 28. *Suppose* $\phi, i : \kappa; \Phi \models \sigma \sqsubseteq \tau$ *and* $i\#\Phi$*. Then for any* $\phi \models S : \kappa$*, we have*

$$\phi; \Phi \models \sigma[S/i] \sqsubseteq \tau[S/i].$$

Proof. By induction on the subtype derivation. For the base cases, we know

$$\forall \phi, i : \kappa.\ (\Phi \Rightarrow R \geq R'),$$

and we need to prove

$$\forall \phi.\ (\Phi \Rightarrow R[S/i] \geq R'[S/i]),$$

but this is clear from the interpretation of R, R'. $\square$

D. The Syntax-Directed system

The syntax-directed system is presented in Figure 8. It works over a uniform context, using $\square$ annotations to "mark", variables not occurring in the original *DFuzz* derivation.

We first prove the system sound with respect the non syntax-directed one.

Lemma 29 (Syntax-directed soundness)**.** *If* $\phi; \Phi \mid \Gamma \vdash_{\mathcal{S}} e : \sigma$ *has a derivation, then* $\phi; \Phi \mid \Gamma \vdash e : \sigma$*.*

Proof. By induction on the derivation proving $\phi; \Phi \mid \Gamma \vdash_{\mathcal{S}} e : \sigma$.

Case: (Var)

$$\frac{}{\phi; \Phi \mid \mathrm{Ectx}(\Gamma^{\bullet}), x :_{[1]} \sigma \vdash_{\mathcal{S}} x : \sigma} \quad (\mathrm{Var})$$

Immediate, the same rule applies.

Case: $(\otimes I)$

$$\frac{\phi; \Phi \mid \Gamma_1 \vdash_{\mathcal{S}} e_1 : \sigma \qquad \phi; \Phi \mid \Gamma_2 \vdash_{\mathcal{S}} e_2 : \tau}{\phi; \Phi \mid \Gamma_1 + \Gamma_2 \vdash_{\mathcal{S}} (e_1, e_2) : \sigma \otimes \tau} \quad (\otimes I)$$

Immediate by induction; the same rule applies.

$$\frac{}{\phi;\Phi \mid \mathrm{Ectx}(\Gamma^\bullet) \vdash_{\mathcal{S}} \mathrm{r} : \mathbb{R}} \quad (\mathrm{Const}_{\mathbb{R}}) \qquad \frac{}{\phi;\Phi \mid \mathrm{Ectx}(\Gamma^\bullet), x :_{[1]} \sigma \vdash_{\mathcal{S}} x : \sigma} \quad (\mathrm{Var})$$

$$\frac{\phi;\Phi \mid \Gamma_1 \vdash_{\mathcal{S}} e_1 : \sigma \qquad \phi;\Phi \mid \Gamma_2 \vdash_{\mathcal{S}} e_2 : \tau}{\phi;\Phi \mid \Gamma_1 + \Gamma_2 \vdash_{\mathcal{S}} (e_1, e_2) : \sigma \otimes \tau} \quad (\otimes I) \qquad \frac{\phi;\Phi \mid \Delta \vdash_{\mathcal{S}} e : \sigma \otimes \tau \qquad \phi;\Phi \mid \Gamma, x :_{[R_1]} \sigma, y :_{[R_2]} \tau \vdash_{\mathcal{S}} e' : \mu}{\phi;\Phi \mid \Gamma + \mathbf{max}(R_{1\square\uparrow}, R_{2\square\uparrow}) \cdot \Delta \vdash_{\mathcal{S}} \mathbf{let}(x,y) = e \mathbf{~in~} e' : \mu} \quad (\otimes E)$$

$$\frac{\phi;\Phi \mid \Gamma_1 \vdash_{\mathcal{S}} e_1 : \sigma \qquad \phi;\Phi \mid \Gamma_2 \vdash_{\mathcal{S}} e_2 : \tau}{\phi;\Phi \mid \mathbf{max}(\Gamma_1, \Gamma_2) \vdash_{\mathcal{S}} \langle e_1, e_2 \rangle : \sigma \mathbin{\&} \tau} \quad (\&\ I) \qquad \frac{\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} e : \sigma_1 \mathbin{\&} \sigma_2}{\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} \pi_i e : \sigma_i} \quad (\&\ E)$$

$$\frac{\phi;\Phi \mid \Gamma, x :_{[R^\bullet]} \sigma \vdash_{\mathcal{S}} e : \tau \qquad \models \forall \phi.\ (\Phi \Rightarrow R \geq R^\bullet{}_{\square\uparrow})}{\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} \lambda(x :_{[R]} \sigma).\ e : !_R \sigma \multimap \tau} \quad (\multimap I) \qquad \frac{\begin{array}{c}\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} e_1 : !_R \sigma \multimap \tau \\ \phi;\Phi \mid \Delta \vdash_{\mathcal{S}} e_2 : \sigma' \qquad \phi;\Phi \models \sigma' \sqsubseteq \sigma\end{array}}{\phi;\Phi \mid \Gamma + R \cdot \Delta \vdash_{\mathcal{S}} e_1\ e_2 : \tau} \quad (\multimap E)$$

$$\frac{\phi, i : \kappa;\Phi \mid \Gamma \vdash_{\mathcal{S}} e : \sigma \qquad i \text{ fresh in } \Phi}{\phi;\Phi \mid \mathbf{sup}(i, \Gamma) \vdash_{\mathcal{S}} \Lambda i : \kappa.\ e : \forall i : \kappa.\ \sigma} \quad (\forall I) \qquad \frac{\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} e : \forall i : \kappa.\ \sigma \qquad \phi \models S : \kappa}{\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} e[S] : \sigma[S/i]} \quad (\forall E)$$

$$\frac{\phi;\Phi \mid \Gamma, x :_{[R]} \sigma \vdash_{\mathcal{S}} e : \sigma' \qquad \phi;\Phi \models \sigma' \sqsubseteq \sigma}{\phi;\Phi \mid \infty \cdot \Gamma \vdash_{\mathcal{S}} \mathbf{fix}\, x : \sigma.\ e : \sigma} \quad (\mathrm{Fix})$$

$$\frac{\begin{array}{c}\phi;\Phi \mid \Delta \vdash_{\mathcal{S}} e : \mathbb{N}[S] \qquad \phi;\Phi, S = 0 \mid \Gamma_0 \vdash_{\mathcal{S}} e_0 : \sigma_0 \\ \phi, i : \mathrm{n};\Phi, S = i+1 \mid \Gamma_s, n :_{[R]} \mathbb{N}[i] \vdash_{\mathcal{S}} e_s : \sigma_s \\ \phi;\Phi, S = 0 \models \sigma_0 \sqsubseteq \sigma \qquad \phi, i : \mathrm{n};\Phi, S = i+1 \models \sigma_s \sqsubseteq \sigma\end{array}}{\phi;\Phi \mid \mathbf{case}(S, i, \Gamma_0, \Gamma_s) + \mathbf{case}(S, i, 0, R_{\square\uparrow}) \cdot \Delta \vdash_{\mathcal{S}} \mathbf{case}\, e\, \mathbf{return}\, \sigma\, \mathbf{of}\ 0 \Rightarrow e_0 \mid n_{[i]} + 1 \Rightarrow e_s : \sigma} \quad (\mathbb{N}\ E)$$

$$\mathrm{Ectx}(\Gamma^\bullet) := \Delta \quad \text{with} \quad \begin{cases} \mathrm{dom}(\Gamma^\bullet) & = \mathrm{dom}(\Delta) \\ \Delta(b) & \equiv \_ :_{\square} \_ \qquad \text{for all } b \in \mathrm{dom}(\Gamma^\bullet) \end{cases}$$

Figure 8. *DFuzz* Type Judgment, Syntax-directed Version

Case: $(\otimes E)$

$$\frac{\phi;\Phi \mid \Delta \vdash_{\mathcal{S}} e : \sigma \otimes \tau \qquad \phi;\Phi \mid \Gamma, x :_{[R_1]} \sigma, y :_{[R_2]} \tau \vdash_{\mathcal{S}} e' : \mu}{\phi;\Phi \mid \Gamma + \mathbf{max}(R_{1\square\uparrow}, R_{2\square\uparrow}) \cdot \Delta \vdash_{\mathcal{S}} \mathbf{let}(x,y) = e \mathbf{~in~} e' : \mu} \quad (\otimes E)$$

By induction, we have

$$\phi;\Phi \mid \Delta \vdash e : \sigma \otimes \tau \quad \text{and} \quad \phi;\Phi \mid \Gamma, x :_{[R_1]} \sigma, y :_{[R_2]} \sigma \vdash e' : \mu$$

By Lemma 27, $\phi;\Phi \models \mathbf{max}(R_{1\square\uparrow}, R_{2\square\uparrow}) \geq R_{i\square\uparrow}$ for $i = 1, 2$. Abbreviating $R^\bullet := \mathbf{max}(R_{1\square\uparrow}, R_{2\square\uparrow})$ and applying weakening we have:

$$\phi;\Phi \mid \Gamma, x :_{[R^\bullet]} \sigma, y :_{[R^\bullet]} \tau \vdash e' : \mu$$

with $R^\bullet \neq \square$ so we have exactly what we need to apply $(\otimes E)$.

Case: $(\&\ I)$

$$\frac{\phi;\Phi \mid \Gamma_1 \vdash_{\mathcal{S}} e_1 : \sigma \qquad \phi;\Phi \mid \Gamma_2 \vdash_{\mathcal{S}} e_2 : \tau}{\phi;\Phi \mid \mathbf{max}(\Gamma_1, \Gamma_2) \vdash_{\mathcal{S}} \langle e_1, e_2 \rangle : \sigma \mathbin{\&} \tau} \quad (\&\ I)$$

By induction, we have

$$\phi;\Phi \mid \Gamma_1 \vdash e_1 : \sigma \quad \text{and} \quad \phi;\Phi \mid \Gamma_2 \vdash e_2 : \tau.$$

By Lemma 27, we have

$$\phi;\Phi \models \mathbf{max}(\Gamma_1, \Gamma_2) \sqsubseteq \Gamma_1 \quad \text{and} \quad \phi;\Phi \models \mathbf{max}(\Gamma_1, \Gamma_2) \sqsubseteq \Gamma_2.$$

By weakening, we can derive

$$\phi;\Phi \mid \mathbf{max}(\Gamma_1, \Gamma_2) \vdash e_1 : \sigma \quad \text{and} \quad \phi;\Phi \mid \mathbf{max}(\Gamma_1, \Gamma_2) \vdash e_2 : \tau,$$

when we can conclude by $(\&\ I)$.

Case: $(\&\ E)$

$$\frac{\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} e : \sigma_1 \mathbin{\&} \sigma_2}{\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} \pi_i e : \sigma_i} \quad (\&\ E)$$

Immediate; the same rule applies.

Case: $(\multimap I)$

$$\frac{\phi;\Phi \mid \Gamma, x :_{[R^\bullet]} \sigma \vdash_{\mathcal{S}} e : \tau \qquad \models \forall \phi.\ (\Phi \Rightarrow R \geq R^\bullet{}_{\square\uparrow})}{\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} \lambda(x :_{[R]} \sigma).\ e : !_R \sigma \multimap \tau} \quad (\multimap I)$$

By induction, we have

$$\phi; \Phi \mid \Gamma, x :_{[R^\bullet]} \sigma \vdash e : \tau$$

and we know $R \neq \Box$ and:

$$\phi; \Phi \models R \geq R^\bullet.$$

By weakening, we have

$$\phi; \Phi \mid \Gamma, x : !_R \sigma \vdash e : \tau,$$

and we can conclude by $(\multimap I)$.

Case: $(\multimap E)$

$$\frac{\begin{array}{c}\phi; \Phi \mid \Gamma \vdash_{\mathcal{S}} e_1 : !_R \sigma \multimap \tau \\ \phi; \Phi \mid \Delta \vdash_{\mathcal{S}} e_2 : \sigma' \qquad \phi; \Phi \models \sigma' \sqsubseteq \sigma\end{array}}{\phi; \Phi \mid \Gamma + R \cdot \Delta \vdash_{\mathcal{S}} e_1 \; e_2 : \tau} \quad (\multimap E)$$

By induction, we have

$$\phi; \Phi \mid \Gamma \vdash e_1 : !_R \sigma \multimap \tau \quad \text{and} \quad \phi; \Phi \mid \Delta \vdash e_2 : \sigma'$$

and we also know

$$\phi; \Phi \models \sigma' \sqsubseteq \sigma.$$

By subtyping on the right, we can derive

$$\phi; \Phi \mid \Delta \vdash e_2 : \sigma,$$

and we can conclude with $(\multimap E)$.

Case: $(\forall I)$

$$\frac{\phi, i : \kappa; \Phi \mid \Gamma \vdash_{\mathcal{S}} e : \sigma \qquad i \text{ fresh in } \Phi}{\phi; \Phi \mid \mathbf{sup}(i, \Gamma) \vdash_{\mathcal{S}} \Lambda i : \kappa.\, e : \forall i : \kappa.\, \sigma} \quad (\forall I)$$

By induction, we have

$$\phi; i : \kappa; \Phi \mid \Gamma \vdash e : \sigma$$

and i fresh in Φ. By Lemma 27, we have

$$\phi; \Phi \models \mathbf{sup}(i, \Gamma) \sqsubseteq \Gamma,$$

and so by weakening, we have

$$\phi, i : \kappa; \Phi \mid \mathbf{sup}(i, \Gamma) \vdash e : \sigma.$$

Now, we can conclude with $(\forall I)$.

Case: $(\forall E)$

$$\frac{\phi; \Phi \mid \Gamma \vdash_{\mathcal{S}} e : \forall i : \kappa.\, \sigma \qquad \phi \models S : \kappa}{\phi; \Phi \mid \Gamma \vdash_{\mathcal{S}} e[S] : \sigma[S/i]} \quad (\forall E)$$

Immediate; the same rule applies.

Case: (Fix)

$$\frac{\phi; \Phi \mid \Gamma, x :_{[R]} \sigma \vdash_{\mathcal{S}} e : \sigma' \qquad \phi; \Phi \models \sigma' \sqsubseteq \sigma}{\phi; \Phi \mid \infty \cdot \Gamma \vdash_{\mathcal{S}} \mathbf{fix}\, x : \sigma.\, e : \sigma} \quad (\text{Fix})$$

By induction; we have

$$\phi; \Phi \mid \Gamma, x : !_R \sigma \vdash e : \sigma'.$$

But we also have $\phi; \Phi \models \sigma' \sqsubseteq \sigma$. By subtyping, we get

$$\phi; \Phi \mid \Gamma, x : !_R \sigma \vdash e : \sigma$$

and we can conclude with (Fix).

Case: $(\mathbb{N}\; E)$

$$\frac{\begin{array}{c}\phi; \Phi \mid \Delta \vdash_{\mathcal{S}} e : \mathbb{N}[S] \qquad \phi; \Phi, S = 0 \mid \Gamma_0 \vdash_{\mathcal{S}} e_0 : \sigma_0 \\ \phi, i : \mathrm{n}; \Phi, S = i + 1 \mid \Gamma_s, n :_{[R]} \mathbb{N}[i] \vdash_{\mathcal{S}} e_s : \sigma_s \\ \phi; \Phi, S = 0 \models \sigma_0 \sqsubseteq \sigma \qquad \phi, i : \mathrm{n}; \Phi, S = i + 1 \models \sigma_s \sqsubseteq \sigma\end{array}}{\phi; \Phi \mid \mathbf{case}(S, i, \Gamma_0, \Gamma_s) + \mathbf{case}(S, i, 0, R_{\Box\uparrow}) \cdot \Delta \vdash_{\mathcal{S}} \mathbf{case}\, e\, \mathbf{return}\, \sigma\, \mathbf{of}\, 0 \Rightarrow e_0 \mid n_{[i]} + 1 \Rightarrow e_s : \sigma} \quad (\mathbb{N}\; E)$$

By induction, we have

$$\begin{aligned}
&\phi; \Phi \mid \Delta \vdash e : \mathbb{N}[S] \\
&\phi; \Phi, S = 0 \mid \Gamma_0 \vdash e_0 : \sigma_0 \\
&\phi, i : \mathrm{n}; \Phi, S = i + 1 \mid \Gamma_s, n : !_R \mathbb{N}[i] \vdash e_s : \sigma_s.
\end{aligned}$$

By Lemma 27, we have

$$\begin{aligned}
&\phi; \Phi, S = 0 \models \mathbf{case}(S, i, \Gamma_0, \Gamma_s) \sqsubseteq \Gamma_0 \\
&\phi, i : \mathrm{n}; \Phi, S = i + 1 \models \mathbf{case}(S, i, \Gamma_0, \Gamma_s) \sqsubseteq \Gamma_s \\
&\phi, i : \mathrm{n}; \Phi, S = i + 1 \models \mathbf{case}(S, i, 0, R_{\Box\uparrow}) \geq R_{\Box\uparrow}
\end{aligned}$$

with $R_{\square\uparrow} \neq \square$, and we also know

$$\phi; \Phi, S = 0 \models \sigma_0 \sqsubseteq \sigma$$
$$\phi, i : \mathrm{n}; \Phi, S = i + 1 \models \sigma_s \sqsubseteq \sigma.$$

By subtyping on the left and right, we have

$$\phi; \Phi \mid \Delta \vdash e : \mathbb{N}[S]$$
$$\phi; \Phi, S = 0 \mid \mathbf{case}(S, i, \Gamma_0, \Gamma_s) \vdash e_0 : \sigma$$
$$\phi, i : \mathrm{n}; \Phi, S = i + 1 \mid \mathbf{case}(S, i, \Gamma_0, \Gamma_s), n : !_{R^\bullet} \mathbb{N}[i] \vdash e_s : \sigma,$$

where $R^\bullet = \mathbf{case}(S, i, 0, R_{\square\uparrow})$. We can then conclude by $(\mathbb{N}\, E)$.

$$\frac{\begin{array}{c}\phi; \Phi \mid \Delta \vdash e : \mathbb{N}[S] \qquad \phi; \Phi, S = 0 \mid \Gamma \vdash e_0 : \sigma \\ \phi, i : \mathrm{n}; \Phi, S = i + 1 \mid \Gamma, n :_{[R]} \mathbb{N}[i] \vdash e_s : \sigma \qquad i \# R \qquad R \neq \square\end{array}}{\phi; \Phi \mid \Gamma + R \cdot \Delta \vdash \mathbf{case}\, e\, \mathbf{return}\, \sigma\, \mathbf{of}\, 0 \Rightarrow e_0 \mid n_{[i]} + 1 \Rightarrow e_s : \sigma} \quad (\mathbb{N}\, E)$$

$\square$

We now prove completeness, that is to say, for every derivation in the original system, the syntax-directed one will have a derivation, possibly even a better from a subtype point of view.

We first need a few auxiliary lemmas:

Lemma 30. *Suppose that $\phi; \Phi \mid \Gamma \vdash_{\mathcal{S}} e : \sigma$ is derivable. Then, for any logically equivalent Ψ such that $\phi \models \Phi \Leftrightarrow \Psi$, there is a derivation of $\phi; \Psi \mid \Gamma \vdash_{\mathcal{S}} e : \sigma$ with the same height.*

Proof. By induction on the derivation. The only place the constraint context is used is when checking constraints of the form

$$\phi; \Phi \models R \geq R'.$$

But since Ψ and Φ are logically equivalent, we evidently have

$$\phi; \Psi \models R \geq R'$$

as well. $\square$

Lemma 31 (Inner Weakening for the Syntax-directed system)**.** *Assume a derivation $\Gamma, x :_{[R]} \sigma \vdash_{\mathcal{S}} e : \tau$, a type σ' such that $\sigma' \sqsubseteq \sigma$. Then, there exists a type τ' and a derivation $\Gamma, x :_{[R]} \sigma' \vdash_{\mathcal{S}} e : \tau'$ such that $\tau' \sqsubseteq \tau$.*

Proof. By induction over the typing derivation. The base cases are immediate. In the induction hypothesis we get to pick the appropriate type and we get a better type in all the cases. $\square$

Lemma 32 (Syntax-directed completeness)**.** *If $\phi; \Phi \mid \Gamma \vdash e : \sigma$ has a derivation, then there exists Γ', σ' such that $\phi; \Phi \mid \Gamma' \vdash_{\mathcal{S}} e : \sigma'$ has a derivation, $\phi; \Phi \models \Gamma \sqsubseteq \Gamma'$, $\phi; \Phi \models \sigma' \sqsubseteq \sigma$.*

Proof. By induction on the derivation proving $\phi; \Phi \mid \Gamma \vdash e : \sigma$.

Case: $(\sqsubseteq .\mathrm{L})$

$$\frac{\phi; \Phi \mid \Delta \vdash e : \sigma \qquad \phi; \Phi \models \Gamma \sqsubseteq \Delta}{\phi; \Phi \mid \Gamma \vdash e : \sigma} \quad (\sqsubseteq .\mathrm{L})$$

Immediate, by induction; the desired context is Δ.

Case: $(\sqsubseteq .\mathrm{R})$

$$\frac{\phi; \Phi \mid \Gamma \vdash e : \sigma \qquad \phi; \Phi \models \sigma \sqsubseteq \tau}{\phi; \Phi \mid \Gamma \vdash e : \tau} \quad (\sqsubseteq .\mathrm{R})$$

Immediate, by induction; the desired subtype is σ.

Case: (Var)

$$\frac{}{\phi; \Phi \mid \Gamma, x :_{[1]} \sigma \vdash x : \sigma} \quad (\mathrm{Var})$$

Immediate; the same rule applies.

Case: $(\otimes I)$

$$\frac{\phi; \Phi \mid \Gamma_1 \vdash e_1 : \sigma \qquad \phi; \Phi \mid \Gamma_2 \vdash e_2 : \tau}{\phi; \Phi \mid \Gamma_1 + \Gamma_2 \vdash (e_1, e_2) : \sigma \otimes \tau} \quad (\otimes I)$$

By induction, we have $\Gamma_1', \Gamma_2', \sigma', \tau'$ such that

$$\phi; \Phi \models \Gamma_1 \sqsubseteq \Gamma_1' \wedge \Gamma_2 \sqsubseteq \Gamma_2' \quad \text{and} \quad \phi; \Phi \models \sigma' \sqsubseteq \sigma \wedge \tau' \sqsubseteq \tau$$

and derivations

$$\phi; \Phi \mid \Gamma_1' \vdash_{\mathcal{S}} e_1 : \sigma' \quad \text{and} \quad \phi; \Phi \mid \Gamma_2' \vdash_{\mathcal{S}} e_2 : \tau'.$$

Then we can conclude by $(\otimes I)$, since Lemma 26 shows

$$\phi; \Phi \models \Gamma_1 + \Gamma_2 \sqsubseteq \Gamma_1' + \Gamma_2' \quad \text{and} \quad \phi; \Phi \models \sigma' \otimes \tau' \sqsubseteq \sigma \otimes \tau.$$

Case: $(\otimes E)$

$$\frac{\phi;\Phi \mid \Delta \vdash e : \sigma \otimes \tau \qquad \phi;\Phi \mid \Gamma, x :_{[R]} \sigma, y :_{[R]} \tau \vdash e' : \mu \qquad R \neq \Box}{\phi;\Phi \mid \Gamma + R \cdot \Delta \vdash \mathbf{let}(x,y) = e \mathbf{~in~} e' : \mu} \quad (\otimes E)$$

By induction and inversion on the subtype relation, we have $\Delta', \Gamma', \sigma', \sigma'', \tau', \tau'', \mu', R_1, R_2$ such that

$$\phi;\Phi \models \Delta \sqsubseteq \Delta'$$
$$\phi;\Phi \models \Gamma, x :_{[R]} \sigma, y :_{[R]} \tau \sqsubseteq \Gamma', x :_{[R_1]} \sigma'', y :_{[R_2]} \tau''$$
$$\phi;\Phi \models \sigma' \sqsubseteq \sigma \wedge \tau' \sqsubseteq \tau$$

this implies $\sigma' \sqsubseteq \sigma''$, $\tau' \sqsubseteq \tau''$, $R \geq R_{1\Box\uparrow}$, and $R \geq R_{2\Box\uparrow}$. We have derivations:

$$\phi;\Phi \mid \Delta' \vdash_{\mathcal{S}} e : \sigma' \otimes \tau' \quad \text{and} \quad \phi;\Phi \mid \Gamma', x :_{[R_1]} \sigma'', y :_{[R_2]} \tau'' \vdash_{\mathcal{S}} e' : \mu'$$

By Lemma 31, we have a derivation:

$$\phi;\Phi \mid \Gamma', x :_{[R_1]} \sigma', y :_{[R_2]} \tau' \vdash_{\mathcal{S}} e' : \mu''$$

with $\mu'' \sqsubseteq \mu'$. Hence, we can produce a syntax-directed derivation now:

$$\phi;\Phi \mid \Gamma' + \mathbf{max}(R'_{1\Box\uparrow}, R'_{2\Box\uparrow}) \cdot \Delta' \vdash_{\mathcal{S}} \mathbf{let}(x,y) = e \mathbf{~in~} e' : \mu''.$$

By Lemma 27, we have that $\phi;\Phi \models R \geq \mathbf{max}(R'_{1\Box\uparrow}, R_{2\Box\uparrow})$ and by Lemma 26,

$$\phi;\Phi \models \Gamma + R \cdot \Delta \sqsubseteq \Gamma' + \mathbf{max}(R'_{1\Box\uparrow}, R_{2\Box\uparrow}) \cdot \Delta',$$

so we are done: the context $\Gamma' + \mathbf{max}(R'_{1\Box\uparrow}, R'_{2\Box\uparrow}) \cdot \Delta'$ and subtype τ'' suffice.

Case: $(\& I)$

$$\frac{\phi;\Phi \mid \Gamma \vdash e_1 : \sigma \qquad \phi;\Phi \mid \Gamma \vdash e_2 : \tau}{\phi;\Phi \mid \Gamma \vdash \langle e_1, e_2 \rangle : \sigma \,\&\, \tau} \quad (\& I)$$

By induction, there exists

$$\phi;\Phi \models \Gamma \sqsubseteq \Gamma'_1 \quad \text{and} \quad \phi;\Phi \models \Gamma \sqsubseteq \Gamma'_2$$
$$\phi;\Phi \models \sigma' \sqsubseteq \sigma \quad \text{and} \quad \phi;\Phi \models \tau' \sqsubseteq \tau$$

such that

$$\phi;\Phi \mid \Gamma'_1 \vdash_{\mathcal{S}} e_1 : \sigma' \quad \text{and} \quad \phi;\Phi \mid \Gamma'_2 \vdash_{\mathcal{S}} e_2 : \tau'.$$

By $(\& I)$, we have

$$\phi;\Phi \mid \mathbf{max}(\Gamma'_1, \Gamma'_2) \vdash_{\mathcal{S}} \langle e_1, e_2 \rangle : \sigma' \,\&\, \tau'.$$

We are done, since by Lemmas 26 and 27,

$$\phi;\Phi \models \sigma' \,\&\, \tau' \sqsubseteq \sigma \,\&\, \tau \quad \text{and} \quad \phi;\Phi \models \Gamma \sqsubseteq \mathbf{max}(\Gamma'_1, \Gamma'_2) \sqsubseteq \Gamma'_i.$$

So, the desired context is $\mathbf{max}(\Gamma'_1, \Gamma'_2)$, and the desired subtype is $\sigma' \,\&\, \tau'$.

Case: $(\& E)$

$$\frac{\phi;\Phi \mid \Gamma \vdash e : \sigma_1 \,\&\, \sigma_2}{\phi;\Phi \mid \Gamma \vdash \pi_i\, e : \sigma_i} \quad (\& E)$$

Immediate, by induction.

Case: $(\multimap I)$

$$\frac{\phi;\Phi \mid \Gamma, x :_{[R]} \sigma \vdash e : \tau \qquad R \neq \Box}{\phi;\Phi \mid \Gamma \vdash \lambda(x :_{[R]} \sigma).e : !_R \sigma \multimap \tau} \quad (\multimap I)$$

By induction, there exists

$$\phi;\Phi \models \Gamma, x :_{[R]} \sigma \sqsubseteq \Gamma', x : !_{R'} \sigma \quad \text{and} \quad \phi;\Phi \models \tau' \sqsubseteq \tau$$

such that

$$\phi;\Phi \mid \Gamma', x :_{[R']} \sigma \vdash_{\mathcal{S}} e : \tau'.$$

By inversion on the subtype relation, we have

$$\phi;\Phi \models R \geq R'_{\Box\uparrow} \wedge \tau' \sqsubseteq \tau.$$

and we are done, since

$$\phi;\Phi \models !_{R'_{\Box\uparrow}} \sigma \multimap \tau' \sqsubseteq !_R \sigma \multimap \tau \quad \text{and} \quad \phi;\Phi \models \Gamma \sqsubseteq \Gamma'.$$

$$\frac{\phi;\Phi \mid \Gamma, x :_{[R^\bullet]} \sigma \vdash_{\mathcal{S}} e : \tau \qquad \models \forall \phi.\, (\Phi \Rightarrow R \geq R^\bullet{}_{\Box\uparrow})}{\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} \lambda(x :_{[R]} \sigma).\, e : !_R \sigma \multimap \tau} \quad (\multimap I)$$

Case: $(\multimap E)$

$$\frac{\phi;\Phi \mid \Gamma \vdash e_1 : !_R\sigma \multimap \tau \qquad \phi;\Phi \mid \Delta \vdash e_2 : \sigma}{\phi;\Phi \mid \Gamma + R \cdot \Delta \vdash e_1\ e_2 : \tau}\ (\multimap E)$$

By induction, there exists $\Gamma', \Delta', R', \sigma', \tau', \sigma''$ such that

$$\phi;\Phi \models \Gamma \sqsubseteq \Gamma'$$
$$\phi;\Phi \models \Delta \sqsubseteq \Delta'$$
$$\phi;\Phi \models !_{R'}\sigma' \multimap \tau' \sqsubseteq !_R\sigma \multimap \tau$$
$$\phi;\Phi \models \sigma'' \sqsubseteq \sigma,$$

and derivations

$$\phi;\Phi \mid \Gamma' \vdash_{\mathcal{S}} e_1 : !_{R'}\sigma' \multimap \tau' \quad \text{and} \quad \phi;\Phi \mid \Delta' \vdash_{\mathcal{S}} e_2 : \sigma''.$$

By inversion on the subtype relation, we have

$$\phi;\Phi \models R \geq R' \quad \text{and} \quad \phi;\Phi \models \sigma'' \sqsubseteq \sigma \sqsubseteq \sigma' \quad \text{and} \quad \phi;\Phi \models \tau' \sqsubseteq \tau.$$

By Lemma 27, the context $\Gamma' + R' \cdot \Delta'$ and subtype τ' suffice.

Case: $(\forall I)$

$$\frac{\phi, i : \kappa;\Phi \mid \Gamma \vdash e : \sigma \qquad i \text{ fresh in } \Phi, \Gamma}{\phi;\Phi \mid \Gamma \vdash \Lambda i : \kappa.\ e : \forall i : \kappa.\ \sigma}\ (\forall I)$$

By induction, there exist

$$\phi, i : \kappa;\Phi \models \sigma' \sqsubseteq \sigma \quad \text{and} \quad \phi, i : \kappa;\Phi \models \Gamma \sqsubseteq \Gamma'$$

such that

$$\phi, i : \kappa;\Phi \mid \Gamma' \vdash_{\mathcal{S}} e : \sigma'.$$

Thus, we have the derivation

$$\phi;\Phi \mid \mathbf{sup}(i, \Gamma') \vdash_{\mathcal{S}} \Lambda i : \kappa.\ e : \forall i : \kappa.\ \sigma'$$

and

$$\phi;\Phi \models \forall i : \kappa.\ \sigma' \sqsubseteq \forall i : \kappa.\ \sigma.$$

By Lemma 27, we actually have

$$\phi;\Phi \models \Gamma \sqsubseteq \mathbf{sup}(i, \Gamma') \sqsubseteq \Gamma',$$

so the context $\mathbf{sup}(i, \Gamma')$ and subtype $\forall i : \kappa.\ \sigma'$ suffices.

Case: $(\forall E)$

$$\frac{\phi;\Phi \mid \Gamma \vdash e : \forall i : \kappa.\ \sigma \qquad \phi \models S : \kappa}{\phi;\Phi \mid \Gamma \vdash e[S] : \sigma[S/i]}\ (\forall E)$$

By induction, there exists

$$\phi;\Phi \models \Gamma \sqsubseteq \Gamma' \quad \text{and} \quad \phi;\Phi \models \forall i : \kappa.\ \sigma' \sqsubseteq \forall i : \kappa.\ \sigma$$

such that

$$\phi;\Phi \mid \Gamma' \vdash_{\mathcal{S}} e : \forall i : \kappa.\ \sigma'.$$

So, we have a derivation

$$\phi;\Phi \mid \Gamma'' \vdash_{\mathcal{S}} e[S/i] : \sigma'[S/i].$$

By Lemma 28,

$$\phi;\Phi \models \sigma'[S/i] \sqsubseteq \sigma[S/i],$$

so the context Γ' and subtype $\sigma'[S/i]$ suffice.

Case: (Fix)

$$\frac{\phi;\Phi \mid \Gamma, x :_{[\infty]} \sigma \vdash e : \sigma}{\phi;\Phi \mid \infty \cdot \Gamma \vdash \mathbf{fix}\, x : \sigma.\ e : \sigma}\ (\text{Fix})$$

By induction, we have

$$\phi;\Phi \models \Gamma, x : !_\infty\sigma \sqsubseteq \Gamma', x : !_R\sigma \quad \text{and} \quad \phi;\Phi \models \sigma' \sqsubseteq \sigma$$

such that

$$\phi;\Phi \mid \Gamma', x : !_R\sigma \vdash_{\mathcal{S}} e : \sigma'.$$

We can then conclude by (Fix): the desired context is $\infty \cdot \Gamma'$ and the desired type is σ.

Case: $(\mathbb{N}\ E)$

$$\frac{\begin{array}{c}\phi;\Phi \mid \Delta \vdash e : \mathbb{N}[S] \qquad \phi;\Phi, S = 0 \mid \Gamma \vdash e_0 : \sigma \\ \phi, i : \mathrm{n};\Phi, S = i+1 \mid \Gamma, n :_{[R]} \mathbb{N}[i] \vdash e_s : \sigma \qquad i \# R \qquad R \neq \Box\end{array}}{\phi;\Phi \mid \Gamma + R \cdot \Delta \vdash \mathbf{case}\, e\, \mathbf{return}\, \sigma\, \mathbf{of}\ 0 \Rightarrow e_0 \mid n_{[i]} + 1 \Rightarrow e_s : \sigma}\ (\mathbb{N}\ E)$$

By induction, there exists

$$\phi;\Phi \models \Delta \sqsubseteq \Delta' \quad \text{and} \quad \phi;\Phi \mid \Delta' \vdash_{\mathcal{S}} e : \mathbb{N}[S'] \quad \text{and} \quad \phi;\Phi \models \mathbb{N}[S'] \sqsubseteq \mathbb{N}[S].$$

By inversion, $\phi;\Phi \models S = S'$. Also by induction,

$$\begin{aligned} &\phi;\Phi, S = 0 \models \Gamma \sqsubseteq \Gamma'_0 \\ &\phi, i : \mathrm{n};\Phi, S = i+1 \models \Gamma, n : !_R\mathbb{N}[i] \sqsubseteq \Gamma'_s, n : !_{R'}\mathbb{N}[i] \\ &\phi;\Phi, S = 0 \models \sigma'_0 \sqsubseteq \sigma \\ &\phi, i : \mathrm{n};\Phi, S = i+1 \models \sigma'_s \sqsubseteq \sigma \end{aligned}$$

such that

$$\begin{aligned} &\phi;\Phi, S = 0 \mid \Gamma'_0 \vdash_{\mathcal{S}} e_0 : \sigma'_0 \\ &\phi, i : \mathrm{n};\Phi, S = i+1 \mid \Gamma'_s, n : !_{R'}\mathbb{N}[i] \vdash_{\mathcal{S}} e_s : \sigma'_s. \end{aligned}$$

By Lemma 30, we also have derivations

$$\begin{aligned} &\phi;\Phi, S' = 0 \mid \Gamma'_0 \vdash_{\mathcal{S}} e_0 : \sigma'_0 \\ &\phi, i : \mathrm{n};\Phi, S' = i+1 \mid \Gamma'_s, n : !_{R'}\mathbb{N}[i] \vdash_{\mathcal{S}} e_s : \sigma'_s \end{aligned}$$

since $\phi;\Phi \models S = S'$.
Hence, we have a derivation

$$\begin{aligned} &\phi;\Phi \mid \mathbf{case}(S', i, \Gamma'_0, \Gamma'_s) + R^\bullet \cdot \Delta' \\ &\vdash_{\mathcal{S}} \mathbf{case}\, e \,\mathbf{return}\, \sigma \,\mathbf{of}\, 0 \Rightarrow e_0 \mid n_{[i]} + 1 \Rightarrow e_s : \sigma, \end{aligned}$$

where $R^\bullet$ is $\mathbf{case}(S', i, 0, {R'}_{\square\uparrow})$. We have

$$\begin{aligned} &\phi;\Phi, S' = 0 \models \mathbf{case}(S', i, \Gamma'_0, \Gamma'_s) \sqsubseteq \Gamma'_0 \\ &\phi, i : \mathrm{n};\Phi, S' = i+1 \models \mathbf{case}(S', i, \Gamma'_0, \Gamma'_s) \sqsubseteq \Gamma'_s \end{aligned}$$

so by Lemma 27

$$\phi;\Phi \models \Gamma \sqsubseteq \mathbf{case}(S', i, \Gamma'_0, \Gamma'_s),$$

and

$$\phi, i : \mathrm{n};\Phi, S' = i+1 \models R \geq R^\bullet \geq {R'}_{\square\uparrow} \quad \text{and} \quad \phi, \Phi \models R \geq R^\bullet$$

thanks to $R \neq \square$.
By weakening, we have

$$\begin{aligned} &\phi;\Phi \mid \Delta' \vdash_{\mathcal{S}} e : \mathbb{N}[S'] \\ &\phi;\Phi, S = 0 \mid \mathbf{case}(S', i, \Gamma'_0, \Gamma'_s) \vdash_{\mathcal{S}} e_0 : \sigma \\ &\phi, i : \mathrm{n};\Phi, S' = i+1 \mid \mathbf{case}(S', i, \Gamma'_0, \Gamma'_s), n : !_{R^\bullet}\mathbb{N}[i] \vdash_{\mathcal{S}} e_s : \sigma, \end{aligned}$$

so we can conlude with ($\mathbb{N}$ E). The context $\mathbf{case}(S', i, \Gamma'_0, \Gamma'_s) + R^\bullet \cdot \Delta'$ and type σ suffice (recall that $\phi;\Phi \models R \geq R^\bullet$, and $\phi;\Phi \models R \cdot \Delta \sqsubseteq R^\bullet \cdot \Delta'$ by Lemma 26).

□

D.1 Algorithm Proofs

Theorem 33 (Algorithmic Soundness)**.** *Suppose $\phi;\Phi;\Gamma^\bullet; e \Longrightarrow \Gamma;\sigma$. Then, there is a derivation of $\phi;\Phi;\Gamma \vdash_{\mathcal{S}} e : \sigma$.*

Proof. By induction on the algorithmic derivations we see that every algorithmic step has an exact correspondence with a syntax-directed derivation. We do a few representative cases:

Case (Var)

$$\frac{}{\phi;\Phi;\Gamma^\bullet, x : \sigma; x \Longrightarrow \mathrm{Ectx}(\Gamma^\bullet), x :_{[1]} \sigma; \sigma} \quad (\mathrm{Var})$$

$$\frac{}{\phi;\Phi \mid \mathrm{Ectx}(\Gamma^\bullet), x :_{[1]} \sigma \vdash_{\mathcal{S}} x : \sigma} \quad (\mathrm{Var})$$

Case ($\multimap E$)

$$\frac{\begin{array}{c}\phi;\Phi;\Gamma^\bullet; e_1 \Longrightarrow \Gamma; !_R\sigma \multimap \tau \\ \phi;\Phi;\Delta^\bullet; e_2 \Longrightarrow \Delta;\sigma' \\ \phi;\Phi \models \sigma' \sqsubseteq \sigma\end{array}}{\phi;\Phi;\Gamma^\bullet; e_1\ e_2 \Longrightarrow \Gamma + R \cdot \Delta; \tau} \quad (\multimap E)$$

$$\frac{\begin{array}{c}\phi;\Phi \mid \Gamma \vdash_{\mathcal{S}} e_1 : !_R\sigma \multimap \tau \\ \phi;\Phi \mid \Delta \vdash_{\mathcal{S}} e_2 : \sigma' \qquad \phi;\Phi \models \sigma' \sqsubseteq \sigma\end{array}}{\phi;\Phi \mid \Gamma + R \cdot \Delta \vdash_{\mathcal{S}} e_1\ e_2 : \tau} \quad (\multimap E)$$

Case $(\otimes E)$

$$\frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Delta; \sigma \otimes \tau \\ \phi; \Phi; \Gamma^\bullet, x : \sigma, y : \tau; e' \Longrightarrow \Gamma, x :_{[R_1]} \sigma, y :_{[R_2]} \tau; \mu\end{array}}{\phi; \Phi; \Gamma^\bullet; \mathbf{let}(x, y) = e \ \mathbf{in} \ e' \Longrightarrow \Gamma + \mathbf{max}(R_{1\Box\uparrow}, R_{2\Box\uparrow}) \cdot \Delta; \mu} \quad (\otimes E)$$

$$\frac{\phi; \Phi \mid \Delta \vdash_{\mathcal{S}} e : \sigma \otimes \tau \qquad \phi; \Phi \mid \Gamma, x :_{[R_1]} \sigma, y :_{[R_2]} \tau \vdash_{\mathcal{S}} e' : \mu}{\phi; \Phi \mid \Gamma + \mathbf{max}(R_{1\Box\uparrow}, R_{2\Box\uparrow}) \cdot \Delta \vdash_{\mathcal{S}} \mathbf{let}(x, y) = e \ \mathbf{in} \ e' : \mu} \quad (\otimes E)$$

□

Theorem 34 (Algorithmic Completeness)**.** *Suppose* $\phi; \Phi; \Gamma \vdash_{\mathcal{S}} e : \sigma$ *is derivable. Then* $\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Gamma; \sigma$.

Proof. By induction on the syntax-directed derivation. The proof is mostly direct, we show a few representative cases.

Case $(\multimap E)$

$$\frac{\begin{array}{c}\phi; \Phi \mid \Gamma \vdash_{\mathcal{S}} e_1 : !_R\sigma \multimap \tau \\ \phi; \Phi \mid \Delta \vdash_{\mathcal{S}} e_2 : \sigma' \qquad \phi; \Phi \models \sigma' \sqsubseteq \sigma\end{array}}{\phi; \Phi \mid \Gamma + R \cdot \Delta \vdash_{\mathcal{S}} e_1 \ e_2 : \tau} \quad (\multimap E)$$

By induction, we have derivations

$$\phi; \Phi; \Gamma^\bullet; e_1 \Longrightarrow \Gamma; !_R\sigma \multimap \tau \quad \text{and} \quad \phi; \Phi; \Delta^\bullet; e_2 \Longrightarrow \Delta; \sigma'.$$

Note that $\Gamma^\bullet = \Delta^\bullet$ for the syntax-directed derivation to be defined, so we can apply the algorithmic rule $(\multimap E)$:

$$\frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet; e_1 \Longrightarrow \Gamma; !_R\sigma \multimap \tau \\ \phi; \Phi; \Delta^\bullet; e_2 \Longrightarrow \Delta; \sigma' \\ \phi; \Phi \models \sigma' \sqsubseteq \sigma\end{array}}{\phi; \Phi; \Gamma^\bullet; e_1 \ e_2 \Longrightarrow \Gamma + R \cdot \Delta; \tau} \quad (\multimap E)$$

Case (Fix)

$$\frac{\phi; \Phi \mid \Gamma, x :_{[R]} \sigma \vdash_{\mathcal{S}} e : \sigma' \qquad \phi; \Phi \models \sigma' \sqsubseteq \sigma}{\phi; \Phi \mid \infty \cdot \Gamma \vdash_{\mathcal{S}} \mathbf{fix}\, x : \sigma.\ e : \sigma} \quad \text{(Fix)}$$

By induction, we have

$$\phi; \Phi; \Gamma^\bullet, x : \sigma; e \Longrightarrow \Gamma, x :_{[R]} \sigma; \sigma'$$

and we can apply the algorithm rule (Fix):

$$\frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet, x : \sigma; e \Longrightarrow \Gamma, x :_{[R]} \sigma; \sigma' \\ \phi; \Phi \models \sigma' \sqsubseteq \sigma\end{array}}{\phi; \Phi; \Gamma^\bullet; \mathbf{fix}\, x : \sigma.\ e : \sigma \Longrightarrow \infty \cdot \Gamma; \sigma} \quad \text{(Fix)}$$

Case $(\otimes E)$

$$\frac{\phi; \Phi \mid \Delta \vdash_{\mathcal{S}} e : \sigma \otimes \tau \qquad \phi; \Phi \mid \Gamma, x :_{[R_1]} \sigma, y :_{[R_2]} \tau \vdash_{\mathcal{S}} e' : \mu}{\phi; \Phi \mid \Gamma + \mathbf{max}(R_{1\Box\uparrow}, R_{2\Box\uparrow}) \cdot \Delta \vdash_{\mathcal{S}} \mathbf{let}(x, y) = e \ \mathbf{in} \ e' : \mu} \quad (\otimes E)$$

We know that $\Gamma^\bullet = \Delta^\bullet$. By induction, we know that:

$$\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Delta; \sigma_1' \otimes \sigma_2'$$

$$\phi; \Phi; \Gamma^\bullet, x_1 : \sigma_1, x_2 : \sigma_2; e' \Longrightarrow \Gamma, x :_{[R_1]} \sigma_1, y :_{[R_2]} \sigma_2; \tau$$

and we know $\phi; \Phi \models \sigma_1' \sqsubseteq \sigma_1 \wedge \sigma_2' \sqsubseteq \sigma_2$, so we apply the algorithmic case $(\otimes E)$:

$$\frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Delta; \sigma \otimes \tau \\ \phi; \Phi; \Gamma^\bullet, x : \sigma, y : \tau; e' \Longrightarrow \Gamma, x :_{[R_1]} \sigma, y :_{[R_2]} \tau; \mu\end{array}}{\phi; \Phi; \Gamma^\bullet; \mathbf{let}(x, y) = e \ \mathbf{in} \ e' \Longrightarrow \Gamma + \mathbf{max}(R_{1\Box\uparrow}, R_{2\Box\uparrow}) \cdot \Delta; \mu} \quad (\otimes E)$$

Case $(\mathbb{N}\ E)$

$$\frac{\begin{array}{c}\phi; \Phi \mid \Delta \vdash_{\mathcal{S}} e : \mathbb{N}[S] \qquad \phi; \Phi, S = 0 \mid \Gamma_0 \vdash_{\mathcal{S}} e_0 : \sigma_0 \\ \phi, i : \mathrm{n}; \Phi, S = i + 1 \mid \Gamma_s, n :_{[R]} \mathbb{N}[i] \vdash_{\mathcal{S}} e_s : \sigma_s \\ \phi; \Phi, S = 0 \models \sigma_0 \sqsubseteq \sigma \qquad \phi, i : \mathrm{n}; \Phi, S = i + 1 \models \sigma_s \sqsubseteq \sigma\end{array}}{\phi; \Phi \mid \mathbf{case}(S, i, \Gamma_0, \Gamma_s) + \mathbf{case}(S, i, 0, R_{\Box\uparrow}) \cdot \Delta \vdash_{\mathcal{S}} \mathbf{case}\, e\, \mathbf{return}\, \sigma\, \mathbf{of}\ 0 \Rightarrow e_0 \mid n_{[i]} + 1 \Rightarrow e_s : \sigma} \quad (\mathbb{N}\ E)$$

We know that $\Gamma^\bullet = \Delta^\bullet$. By induction, we know that:

$$\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Delta; \mathbb{N}[S]$$

$$\phi; \Phi, S = 0; \Gamma^\bullet; e_0 \Longrightarrow \Gamma_0; \sigma_0$$

$$\phi, i : \mathrm{n}; \Phi, S = i + 1; \Gamma^\bullet, x : \mathbb{N}[i]; e_s \Longrightarrow \Gamma_s, x :_{[R']} \mathbb{N}[i]; \sigma_s$$

and we know

$$\phi; \Phi, S = 0 \models \sigma_0 \sqsubseteq \sigma \quad \text{and} \quad \phi, i : \mathrm{n}; \Phi, S = i + 1 \models \sigma_s \sqsubseteq \sigma.$$

We can conclude with the algorithmic rule $(\mathbb{N}\ E)$:

$$\frac{\begin{array}{c}\phi; \Phi; \Gamma^\bullet; e \Longrightarrow \Delta; \mathbb{N}[S] \qquad \phi; \Phi, S = 0; \Gamma^\bullet; e_0 \Longrightarrow \Gamma_0; \sigma_0 \\ \phi, i : \mathrm{n}; \Phi, S = i + 1; \Gamma^\bullet, x : \mathbb{N}[i]; e_s \Longrightarrow \Gamma_s, x :_{[R']} \mathbb{N}[i]; \sigma_s \\ \phi; \Phi, S = 0 \models \sigma_0 \sqsubseteq \sigma \qquad \phi, i : \mathrm{n}; \Phi, S = i + 1 \models \sigma_s \sqsubseteq \sigma\end{array}}{\begin{array}{c}\phi; \Phi; \Gamma^\bullet; \mathbf{case}\, e\, \mathbf{return}\, \sigma\, \mathbf{of}\, 0 \mapsto e_0 \mid x_{[i]} + 1 \mapsto e_s \\ \Longrightarrow \mathbf{case}(S, \Gamma_0, i, \Gamma_s) + \mathbf{case}(S, 0, i, R'_{\Box\uparrow}) \cdot \Delta; \sigma\end{array}} \ (\mathbb{N}\ E)$$

□

E. Auxiliary Lemmas

Lemma 35 (Standard Annotations)**.** *Assume annotations in a term e range over regular sensitivities and $\phi; \Phi \mid \Gamma \vdash_{\mathcal{S}} e : \sigma$. Then:*

- *σ has no extended sensitivities; and*
- *all the constraints are of the form $\models \forall \phi.\ (\Phi \Rightarrow R \geq R')$ where R is a standard sensitivity term.*

This directly implies Lemma 11.

Proof. The first point is clear by inspecting the rules in Figure 8: by induction, the type of any expression has only regular sensitivities. The second point is also clear: in all subtype checks in Figure 8, both types have no extended sensitivities by the first point. The only place where we check against an extended sensitivity is in rule $(\multimap I)$, with constraint

$$\models \forall \phi.\ (\Phi \Rightarrow R \geq R').$$

Here, the R is a standard sensitivity term since it is an annotation, but the R' may be an extended sensitivity. □